```
          Exist=1; break;
        end
      end
      if(Exist==0)
        FVAllMat=[FVAllMat;VShared]; FNumVAllMat=[FNumVAllMat;NumShared];
      end
    end
  end
end
FVMat=[]; FNumVMat=[]; FVCFMat=[]; FNumVCFMat=[];
for i=1:NumFVAllMat,
  VThisFace=find(FVAllMat(i,:)'); NumVThisFace=size(VThisFace,1);
  IncludeMe=1; IncludeMeToo=1;
  for j=1:NumVThisFace,
    if(VFrame(VThisFace(j,1),1)==0)
      IncludeMe=0;
    end
    if(VCFrame(VThisFace(j,1),1)==0)
      IncludeMeToo=0;
    end
  end
  if(IncludeMe==1)
    FVMat=[FVMat;FVAllMat(i,:)]; FNumVMat=[FNumVMat;FNumVAllMat(i,:)];
  end
  if(IncludeMeToo==1)
    FVCFMat=[FVCFMat;FVAllMat(i,:)]; FNumVCFMat=[FNumVCFMat;FNumVAllMat(i,:)];
  end
end
NumFVMat=size(FVMat,1); NumFVCFMat=size(FVCFMat,1); FDim=Dimension-1;
fprintf(pt3,'Face dimension: %d\n',FDim); fprintf(pt3,'Number of faces: %d\n',NumFVMat);
fprintf(pt3,'Number of vertices: \n [');
for i=1:NumFVMat,
  fprintf(pt3,'%d ',FNumVMat(i,1));
  if(mod(i,10)==0)
    fprintf(pt3,'...\n');
  end
end
fprintf(pt3,']\n'); fprintf(pt3,'Number of faces of nice cells: %d\n',NumFVCFMat);
fprintf(pt3,'Number of vertices: \n [');
for i=1:NumFVCFMat,
  fprintf(pt3,'%d ',FNumVCFMat(i,1));
  if(mod(i,10)==0)
    fprintf(pt3,'...\n');
  end
end
fprintf(pt3,']\n'); DVMat=FVCFMat; NumD=NumFVCFMat;
for d=3:Dimension,
  FaceCond=Dimension-d+2; DNeighDDMat=sparse(NumD,NumD); dVMat=[]; dNumVMat=[];
  for i=1:(NumD-1),
    for j=(i+1):NumD,
      VShared=and(DVMat(i,:), DVMat(j,:)); NumShared =sum(VShared, 2);
      NumdVMat=size(dVMat,1);
      if (NumShared >= FaceCond)
        DNeighDDMat(i,j) =1; DNeighDDMat(j,i) =1; Exist=0;
        for k=1:NumdVMat,
          MatchExistingdV=sum(and(VShared,dVMat(k,:)),2);
          if(MatchExistingdV>=FaceCond)
            Exist=1; break;
          end
        end
        if(Exist==0)
          dVMat=[dVMat;VShared]; dNumVMat=[dNumVMat;NumShared];
        end
      end
    end
  end
  FDim=Dimension-d+1; fprintf(pt3,'Face dimension: %d\n',FDim);
  fprintf(pt3,'Number of faces: %d\n',NumdVMat);
  if(FDim~=1)
    fprintf(pt3,'Number of vertices: \n [');
    for i=1:NumdVMat,
      fprintf(pt3,'%d ',dNumVMat(i,1));
      if(mod(i,10)==0)
        fprintf(pt3,'...\n');
      end
    end
    fprintf(pt3,']\n');
  end
  DVMat=dVMat; NumD=NumdVMat;
  if(FDim==2)
    FVMat=DVMat;
  end
```

```
end
Time=cputime-t; NumNiceC=sum(CFrame); NumVBound=sum(VFrame);
```

§ A.15 Beam intersection program

```
% penc.m     by K N J Tiyapan, 15th July, 2001
clear all; format long g; format compact; NumCell=1000; rand('state',sum(100*clock));
X=1.5*rand(NumCell,1)-0.25*ones(NumCell,1); Y=1.5*rand(NumCell,1)-0.25*ones(NumCell,1);
% X=poissrnd(.5,NumCell,1); Y=poissrnd(.5,NumCell,1);
% X=raylrnd([1:NumCell])'; Y=raylrnd([1:NumCell])';
% Max=0.8*max([X;Y]); X=X/Max; Y=Y/Max;
[XVoro,YVoro]=voronoi(X,Y); SizeV=size(XVoro,2); Xv=[]; Yv=[];
for j=1:SizeV,
  if((((XVoro(1,j)>0) & (XVoro(1,j)<1)) & ((YVoro(1,j)>0) & (YVoro(1,j)<1))) | ...
   (((XVoro(2,j)>0) & (XVoro(2,j)<1)) & ((YVoro(2,j)>0) & (YVoro(2,j)<1))))
    Xv=[Xv,XVoro(:,j)]; Yv=[Yv,YVoro(:,j)];
  end
end
clf; plot(Xv,Yv); V1=[Xv(1,:);Yv(1,:)]'; V2=[Xv(2,:);Yv(2,:)]'; NumE=size(V1,1);
axis equal; axis([0 1 0 1]); Slope=2; Const=-.5; Cx=-.1; Dx=1.1;
Cy=Slope*Cx+Const; Dy=Slope*Dx+Const; C=[Cx,Cy]; D=[Dx,Dy]; CD=D-C; DistVect=[];
hold on; plot([C(1,1);D(1,1)],[C(1,2);D(1,2)]);
for i=1:NumE,
  A=V1(i,:); B=V2(i,:); AB=B-A; CA=A-C; Denom=det([AB;CD]);
  RNom=det([CD;CA]); SNom=det([AB;CA]);
  if(Denom~=0)
    r=RNom/Denom; s=SNom/Denom;
    if((r<=1) & (r>=0) & (s>=0) & (s<=1))
      P=A+r*AB; hold on; plot(P(1,1),P(1,2),'.','lineWidth',2);
      CP=P-C; Distance=sqrt(CP(1,1)*CP(1,1)+CP(1,2)*CP(1,2));
      DistVect=[DistVect;Distance];
    end
  end
end
SortDist=sort(DistVect); Dist=[]; NumSortDist=size(SortDist,1);
for i=2:NumSortDist,
  Dist=[Dist;(SortDist(1,1)-SortDist(i,1))];
end
Dist=abs(Dist); NumDist=size(Dist,1); PairDist=[];
for i=2:NumDist,
   PairDist=[PairDist;(Dist(i,1)-Dist((i-1),1))];
end
Xc=[]; Yc=[];
for i=1:NumCell,
  if((X(i,1)>0) & (X(i,1)<1) & (Y(i,1)>0) & (Y(i,1)<1))
    Xc=[Xc;X(i,1)]; Yc=[Yc;Y(i,1)];
  end
end
NumC=size(Xc,1); NormBase=1/sqrt(NumC); format short g; NumE; NumC;
NumEExcess=size(XVoro,2); MeanPairDist=mean(PairDist); VarPairDist=var(PairDist);
PairDistMNorm=PairDist/MeanPairDist; MeanPairDistMNorm=mean(PairDistMNorm);
VarPairDistMNorm=var(PairDistMNorm); Moment2PairDistMNorm=moment(PairDistMNorm,2);
Moment3PairDistMNorm=moment(PairDistMNorm,3); PairDistNNorm=PairDist/NormBase;
MeanPairDistNNorm=mean(PairDistNNorm); VarPairDistNNorm=var(PairDistNNorm);
Moment2PairDistNNorm=moment(PairDistNNorm,2); Moment3PairDistNNorm=moment(PairDistNNorm,3);
```

§ A.16 Number of vertices in high dimensions

```
% vhd.m ,aka f69.m, (c) K. N. Tiyapan 25th March, 2001
clear; more off; format long g; echo off; TimeStartPreparing =cputime;
pt1 =fopen('/home/mjkvjkt/vn/v69','r'); sc1 =fscanf(pt1, '%d', 4);
Dimension =sc1(1,1); VerticeNum =sc1(2,1); CellNum =sc1(3,1);
sc2 =fscanf(pt1, '%f', [Dimension, VerticeNum]); Vertices =sc2'; Framed =ones(CellNum, 1);
for i=1:CellNum,
  sc3 =fscanf(pt1, '%d', 1);
  for j=1:sc1,
    sc4 =fscanf(pt1, '%d', 1); Num =sc4+1;
    if ( max(abs(Vertices(Num, :))) > 0.5 )
      Framed(i,1) =0;
    end
  end
end
fclose(pt1); pt2 =fopen('/home/mjkvjkt/vn/n69','r');
sc5=fscanf(pt2, '%d', CellNum); VerticesPerCell =sc5; fclose(pt2);
MinVPerCell =min(VerticesPerCell); MaxVPerCell =max(VerticesPerCell);
MeanVPerCell =mean(VerticesPerCell); ScndMVPerCell =moment(VerticesPerCell,2);
ThrdMVPerCell =moment(VerticesPerCell,3); FrthMVPerCell =moment(VerticesPerCell,4);
VarVPerCell =var(VerticesPerCell); StdVPerCell =std(VerticesPerCell);
GMeanVPerCell =geomean(VerticesPerCell); HMeanVPerCell =harmmean(VerticesPerCell);
MedVPerCell =median(VerticesPerCell); MadVPerCell =mad(VerticesPerCell);
KurVPerCell =kurtosis(VerticesPerCell); TabVPerCell =tabulate(VerticesPerCell);
VPerInnerCell =frameit(VerticesPerCell,Framed); InnerVNum =sum(Framed);
MinVPerInnerCell =min(VPerInnerCell); MaxVPerInnerCell =max(VPerInnerCell);
MeanVPerInnerCell =mean(VPerInnerCell); ScndMVPerInnerCell =moment(VPerInnerCell,2);
ThrdMVPerInnerCell =moment(VPerInnerCell,3); FrthMVPerInnerCell =moment(VPerInnerCell,4);
VarVPerInnerCell =var(VPerInnerCell); StdVPerInnerCell =std(VPerInnerCell);
GMeanVPerInnerCell =geomean(VPerInnerCell); HMeanVPerInnerCell =harmmean(VPerInnerCell);
MedVPerInnerCell =median(VPerInnerCell); MadVPerInnerCell =mad(VPerInnerCell);
KurVPerInnerCell =kurtosis(VPerInnerCell); TabVPerInnerCell =tabulate(VPerInnerCell);
more on; clf; bar(TabVPerCell(:,1),TabVPerCell(:,2));
```

§ A.17 Example batch program, simulation and data extraction

```
/home/bin/rbox 1000 t3765098 D6|/home/bin/qhull v o>/home/qhull/wrk/v761
/usr/bin/tail -1000 /home/qhull/wrk/v761|/usr/bin/cut -f1 -d" ">/home/qhull/wrk/n761
/usr/bin/rm -f /home/qhull/wrk/v761
/usr/local/bin/matlab < /home/qhull/wrk/f761.m
```

§ A.18 Voronoi operator of various order

```
% vov.m, voronoi of voronoi, (c) Kit Tiyapan, 2002
clear all; itn=6; can=100;
rand('state',sum(100*clock));
x{1}=rand(can,2);
for m=1:itn,
  [va{m},ca{m}]=voronoin(x{m}); van(m)=size(va{m},1);
  vin{m}=ones(1,van(m)); vin{m}(1)=0;
  for i=2:van(m),
    if((max(va{m}(i,:))>1)|(min(va{m}(i,:))<0))
      vin{m}(i)=0;
    end
  end
  c{m}=[]; cnt=0;
  for i=1:can(m),
    ca{m}{i,2}=size(ca{m}{i,1},2); in=1;
    for j=1:ca{m}{i,2},
      if(~vin{m}(ca{m}{i,1}(j)))
        in=0; break;
      end
    end
    if(in)
      cnt=cnt+1; c{m}{cnt,1}=ca{m}{i,1}; c{m}{cnt,2}=ca{m}{i,2};
    end
  end
  cn(m)=size(c{m},1); cnt=0;
  for i=1:van(m),
    if(vin{m}(i))
      cnt=cnt+1; vin{m}(i)=cnt;
    end
  end
  for i=1:cn(m),
    for j=1:c{m}{i,2},
      c{m}{i,1}(j)=vin{m}(c{m}{i,1}(j));
    end
  end
  v{m}=[];
  for i=1:van(m),
    if(vin{m}(i))
      v{m}=[v{m};va{m}(i,:)];
    end
  end
  vn(m)=size(v{m},1); x{m+1}=v{m}; can(m+1)=vn(m); figure(m); clf; hold on;
  for i=1:cn(m),
    tmp=[c{m}{i,1},c{m}{i,1}(1)];
    for j=1:c{m}{i,2},
        plot([v{m}(tmp(j),1),v{m}(tmp(j+1),1)],[v{m}(tmp(j),2),v{m}(tmp(j+1),2)]);
    end
  end
  axis equal; axis off;
end
```

§ A.19 Voronoi data structure for filtering membrane study

```
% vff.m, Voronoi for filtration, (c) 2002, Kit Tiyapan. 10th Nov.
clear all; can=100; rand('state',sum(100*clock)); ca=rand(can,3);
[va,vca]=voronoin(ca); van=size(va,1); vin=zeros(1,van);
for i=1:van,
  vin(i)=(min(va(i,:))>0) & (max(va(i,:))<1);
end
cin=ones(1,can);
for i=1:can,
  vca{i,2}=vca{i,1}; vca{i,1}=size(vca{i,2},2);
  for j=1:vca{i,1},
    if(~vin(vca{i,2}(j)))
      cin(i)=0; break;
    end
  end
end
c=[]; vc=[]; cnt=0; v=[]; cnu=0; vin=sparse(1,van);
for i=1:can,
  if(cin(i))
    cnt=cnt+1; cin(i)=cnt; c(cnt,:)=ca(i,:); vc{cnt,1}=vca{i,1}; vc{cnt,2}=vca{i,2};
    for j=1:vc{cnt,1},
      if(~vin(vca{i,2}(j)))
        cnu=cnu+1; vin(vca{i,2}(j),1)=cnu; v(cnu,:)=va(vca{i,2}(j),:);
      end
    end
    for j=1:vc{cnt,1},
      vc{cnt,2}(j)=vin(vca{i,2}(j));
    end
  end
end
cn=cnt; vn=cnu;
for i=1:cn,
  tmp=ones(size(vc{i,2})); vc{i,3}=sparse(tmp,vc{i,2},tmp,1,vn);
end
ta=delaunayn(ca); tan=size(ta,1);
t=[]; cnt=0; bdr=[]; cnu=0;
for i=1:tan,
  in=1;
  for j=1:4,
    if(~cin(ta(i,j)))
      in=0; break;
    end
  end
  if(in)
    cnt=cnt+1;
    t(cnt,:)=ta(i,:);
    for j=1:4,
      t(cnt,j)=cin(ta(i,j));
    end
  else
    for j=1:3,
      for k=(j+1):4,
        if((cin(ta(i,j))) | (cin(ta(i,k))))
          cnu=cnu+1; tma=vca{ta(i,j),2}; tmb=vca{ta(i,k),2};
          tmd=ones(1,vca{ta(i,j),1}); tme=ones(1,vca{ta(i,k),1});
          tmp=find(sparse(tmd,tma,tmd,1,van) & sparse(tme,tmb,tme,1,van)); tmq=[];
          if(cin(ta(i,j)))
            bdr{cnu,1}=cin(ta(i,j));
          else
            bdr{cnu,1}=cin(ta(i,k));
          end
          bdr{cnu,2}=size(tmp,2);
          for l=1:bdr{cnu,2},
            tmq=[tmq,vin(tmp(l))];
          end
          bdr{cnu,3}=tmq;
        end
      end
    end
  end
end
tn=cnt; bdrn=size(bdr,1); bcc=sparse(cn,cn);
for i=1:tn,
  for j=1:3,
    for k=(j+1):4,
      bcc(t(i,j),t(i,k))=1; bcc(t(i,k),t(i,j))=1;
    end
  end
end
[tmf,tmg]=find(triu(bcc)); bn=size(tmf,1); b=[];
```

```
for i=1:bn,
  b{i,1}(1,:)=[tmf(i),tmg(i)]; b{i,4}=vc{tmf(i),3} & vc{tmg(i),3};
  [nth,b{i,3}]=find(b{i,4}); b{i,2}=size(b{i,3},2); tmp=[];
  for j=1:b{i,2},
    tmp=[tmp;v(b{i,3}(j),:)];
  end
  tma=max(tmp,[],1)-min(tmp,[],1); tmb=sortrows([1,2,3;tma]',2);
  tmb=round(tmb(:,1)); tmq=[tmp(:,tmb(1,1)),tmp(:,tmb(2,1))];
  tmp=delaunay(tmq(:,1),tmq(:,2)); tmq=sparse(b{i,2},b{i,2});
  for j=1:size(tmp,1),
    for k=1:2,
      for l=(k+1):3,
        tmq(tmp(j,k),tmp(j,l))=tmq(tmp(j,k),tmp(j,l))+1;
        tmq(tmp(j,l),tmp(j,k))=tmq(tmp(j,l),tmp(j,k))+1;
      end
    end
  end
  [tma,tmb,tmc]=find(tmq); tmq=sparse(vn,vn);
  for j=1:size(tma,1),
    if(~(tmc(j)-1))
      tmq(b{i,3}(tma(j)),b{i,3}(tmb(j)))=1; tmq(b{i,3}(tmb(j)),b{i,3}(tma(j)))=1;
    end
  end
  tma=[b{i,3}(1)]; tmb=b{i,3}(1); [nth,tmc]=find(tmq(tmb,:));
  tmd=tmc(1); tma=[tma,tmd]; tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  while(tmb-b{i,3}(1))
    [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1); tma=[tma,tmd];
    tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  end
  b{i,6}=tma; b{i,5}=b{i,6}(1,1:b{i,2}); b{i,7}=2;
end
for i=1:bn,
  bcc(b{i,1}(1),b{i,1}(2))=i; bcc(b{i,1}(2),b{i,1}(1))=i;
end
bc=[];
for i=1:cn,
  bc{i,1}=[];
end
for i=1:bn,
  bc{b{i,1}(1),1}=[bc{b{i,1}(1)},i]; bc{b{i,1}(2),1}=[bc{b{i,1}(2)},i];
end
n=bn;
for i=1:bdrn,
  n=n+1; b{n,1}=bdr{i,1}; b{n,2}=bdr{i,2}; b{n,3}=bdr{i,3};
  tmp=ones(1,bdr{i,2}); b{n,4}=sparse(tmp,bdr{i,3},tmp,1,vn); tmp=[];
  for j=1:b{n,2},
    tmp=[tmp;v(b{n,3}(j),:)];
  end
  tma=max(tmp,[],1)-min(tmp,[],1); tmb=sortrows([1,2,3;tma]',2);
  tmb=round(tmb(:,1)); tmq=[tmp(:,tmb(1,1)),tmp(:,tmb(2,1))];
  tmp=delaunay(tmq(:,1),tmq(:,2)); tmq=sparse(b{n,2},b{n,2});
  for j=1:size(tmp,1),
    for k=1:2,
      for l=(k+1):3,
        tmq(tmp(j,k),tmp(j,l))=tmq(tmp(j,k),tmp(j,l))+1;
        tmq(tmp(j,l),tmp(j,k))=tmq(tmp(j,l),tmp(j,k))+1;
      end
    end
  end
  [tma,tmb,tmc]=find(tmq); tmq=sparse(vn,vn);
  for j=1:size(tma,1),
    if(~(tmc(j)-1))
      tmq(b{n,3}(tma(j)),b{n,3}(tmb(j)))=1; tmq(b{n,3}(tmb(j)),b{n,3}(tma(j)))=1;
    end
  end
  tma=[b{n,3}(1)]; tmb=b{n,3}(1); [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1);
  tma=[tma,tmd]; tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  while(tmb-b{n,3}(1))
    [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1); tma=[tma,tmd];
    tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  end
  b{n,6}=tma; b{n,5}=b{n,6}(1,1:b{n,2}); b{n,7}=1;
end
for i=(bn+1):n,
  bc{b{i,1}}=[];
end
for i=(bn+1):n,
  bc{b{i,1},1}=[bc{b{i,1}},i];
end
% for graphical tests
clf; hold on;
for k=1:cn,
  [nth,ntg,tma]=find(bcc(k,:));
  for i=1:size(tma,2),
    tmp=[];
    for j=1:(b{tma(i),2}+1),
      tmp=[tmp;v(b{tma(i),6}(j),:)];
    end
    plot3(tmp(:,1),tmp(:,2),tmp(:,3));
  end
end
for i=bn:n,
     tmp=[];
    for j=1:(b{i,2}+1),
      tmp=[tmp;v(b{i,6}(j),:)];
    end
    plot3(tmp(:,1),tmp(:,2),tmp(:,3));
end
axis equal; axis off; rotate3d;
```

§ A.20 Centroid process on 2-d VT

```
% ctc.m, test centroid of polygons. (c) Kit Tiyapan, 2002.
clear all; rand('state',sum(100*clock)); can=200; ca=rand(can,2);
for z=1:3,
  [va,vca]=voronoin(ca); van=size(va,1); vcan=size(vca,1); vin=zeros(1,van);
  for i=1:van,
    if((max(va(i,:))<1) & (min(va(i,:)>0)))
      vin(i)=1;
    end
  end
  cin=zeros(1,vcan);
  for i=1:vcan,
    vca{i,2}=vca{i}; vca{i,1}=size(vca{i,2},2); in=1;
    for j=1:vca{i,1},
      if(~vin(vca{i,2}(j)))
        in=0; break;
      end
    end
    if(in)
      cin(i)=1;
    end
  end
  vin=zeros(1,van);
  for i=1:vcan,
    if(cin(i))
      for j=1:vca{i,1},
        vin(vca{i,2}(j))=1;
      end
    end
  end
  v=[]; cnt=0;
  for i=1:van,
    if(vin(i))
      cnt=cnt+1; vin(i)=cnt; v(cnt,:)=va(i,:);
    end
  end
  vn=cnt; c=[]; vc=[]; cnt=0;
  for i=1:vcan,
    if(cin(i))
      cnt=cnt+1; c(cnt,:)=ca(i,:); vc{cnt,1}=vca{i,1}; vc{cnt,2}=vca{i,2};
    end
  end
  cn=cnt;
  for i=1:cn,
    for j=1:vc{i,1},
      vc{i,2}(j)=vin(vc{i,2}(j));
    end
  end
  for i=1:cn,
    tmp=[];
    for j=1:vc{i,1},
      tmp=[tmp;v(vc{i,2}(j),:)];
    end
    d=delaunay(tmp(:,1),tmp(:,2)); dn=size(d,1); tmj=[]; tmk=[]; tmi=0;
    for j=1:dn,
      tma=[v(vc{i,2}(d(j,1)),:);v(vc{i,2}(d(j,2)),:);v(vc{i,2}(d(j,3)),:)];
      tmp=sum(tma)/3; tma=[tma;tma(1,:)]; tmb=[];
      for k=1:3,
        tmb=[tmb,sqrt(sum((diff(tma(k:(k+1),:),1,1)).^2))];
      end
      tmc=sum(tmb)/2; tmq=sqrt(tmc*(tmc-tmb(1))*(tmc-tmb(2))*(tmc-tmb(3)));
      tmk=[tmk;tmq]; tmi=tmi+tmq*tmp;
    end
    vc{i,3}=sum(tmk); vc{i,4}=tmi/sum(tmk);
  end
  vs{z,1}=c; vs{z,2}=v; vs{z,3}=vc; g=[];
  for i=1:cn,
    g=[g;vc{i,4}];
  end
  ca=g; figure(z); clf; hold on;
  for i=1:cn,
    tmp=[];
    for j=1:vc{i,1},
      tmp=[tmp;v(vc{i,2}(j),:)];
    end
    tmp=[tmp;tmp(1,:)]; plot(tmp(:,1),tmp(:,2));
  end
  for i=1:cn,
    plot(vc{i,4}(1),vc{i,4}(2),'o');
  end
  for i=1:cn,
    plot(c(i,1),c(i,2),'.');
  end
  axis equal; axis off;
end
```

§ A.21 Centroid process on 3-d VT

```
% cgi.m, c.g. operator on 3-d Voronoi, Kit Tiyapan (c) 12th November 2002.
clear all; can=400; rand('state',sum(100*clock)); ca=rand(can,3);
for z=1:3,
  cmx=max(max(ca)); cmn=min(min(ca)); [va,vca]=voronoin(ca);
  van=size(va,1); vin=zeros(1,van);
  for i=1:van,
    vin(i)=(min(va(i,:))>cmn) & (max(va(i,:))<cmx);
  end
  cin=ones(1,can);
  for i=1:can,
    vca{i,2}=vca{i,1}; vca{i,1}=size(vca{i,2},2);
    for j=1:vca{i,1},
      if(~vin(vca{i,2}(j)))
        cin(i)=0; break;
      end
    end
  end
  c=[];
  vc=[]; cnt=0; v=[]; cnu=0; vin=sparse(1,van);
  for i=1:can,
    if(cin(i))
      cnt=cnt+1; cin(i)=cnt; c(cnt,:)=ca(i,:); vc{cnt,1}=vca{i,1}; vc{cnt,2}=vca{i,2};
      for j=1:vc{cnt,1},
        if(~vin(vca{i,2}(j)))
          cnu=cnu+1; vin(vca{i,2}(j),1)=cnu; v(cnu,:)=va(vca{i,2}(j),:);
        end
      end
      for j=1:vc{cnt,1},
        vc{cnt,2}(j)=vin(vca{i,2}(j));
      end
    end
  end
  cn=cnt; vn=cnu;
  for i=1:cn,
    tmp=ones(size(vc{i,2})); vc{i,3}=sparse(tmp,vc{i,2},tmp,1,vn);
  end
  ta=delaunayn(ca); tan=size(ta,1); t=[]; cnt=0; bdr=[]; cnu=0;
  tmt=sparse(1,vn); tmm=sparse(can,can);
  for i=1:tan,
    in=1;
    for j=1:4,
      if(~cin(ta(i,j)))
        in=0; break;
      end
    end
    if(in)
      cnt=cnt+1; t(cnt,:)=ta(i,:);
      for j=1:4,
        t(cnt,j)=cin(ta(i,j));
      end
    else
      for j=1:3,
        for k=(j+1):4,
        tma=ta(i,j); tmb=ta(i,k);
        if(((cin(tma)) | (cin(tmb))) & ~tmm(tma,tmb))
          tmm(tma,tmb)=1; tmm(tmb,tma)=1; tma=vca{ta(i,j),2}; tmb=vca{ta(i,k),2};
          tmd=ones(1,vca{ta(i,j),1}); tme=ones(1,vca{ta(i,k),1});
          tmp=find(sparse(tmd,tma,tmd,1,van) & sparse(tme,tmb,tme,1,van));
          tmq=[]; tmn=size(tmp,2);
          for l=1:tmn,
            tmq=[tmq,vin(tmp(l))];
          end
          tma=ones(1,tmn); tmb=sparse(tma,tmq,tma,1,vn); xst=0;
          for m=1:cnu,
            if(~(tmn-bdr{m,2}))
              tmc=tmb & tmt(m,:);
              if(min(tmc))
                xst=1; break;
              end
            end
            if(xst)
              break;
            end
          end
          if(~xst)
            cnu=cnu+1;
            if(cin(ta(i,j)))
              bdr{cnu,1}=cin(ta(i,j));
            else
              bdr{cnu,1}=cin(ta(i,k));
            end
            bdr{cnu,2}=tmn; bdr{cnu,3}=tmq; bdr{cnu,4}=tmb; tmt(cnu,:)=tmb;
          end
        end
      end
    end
  end
end
tn=cnt; bdrn=size(bdr,1); bcc=sparse(cn,cn);
for i=1:tn,
  for j=1:3,
    for k=(j+1):4,
      bcc(t(i,j),t(i,k))=1; bcc(t(i,k),t(i,j))=1;
    end
  end
end
[tmf,tmg]=find(triu(bcc)); bn=size(tmf,1); b=[];
for i=1:bn,
  b{i,1}(1,:)=[tmf(i),tmg(i)]; b{i,4}=vc{tmf(i),3} & vc{tmg(i),3};
  [nth,b{i,3}]=find(b{i,4}); b{i,2}=size(b{i,3},2); tmp=[];
  for j=1:b{i,2},
    tmp=[tmp;v(b{i,3}(j),:)];
  end
  tma=max(tmp,[],1)-min(tmp,[],1); tmb=sortrows([1,2,3;tma]',2);
  tmb=round(tmb(:,1)); tmq=[tmp(:,tmb(1,1)),tmp(:,tmb(2,1))];
  tmp=delaunay(tmq(:,1),tmq(:,2)); tmq=sparse(b{i,2},b{i,2});
  for j=1:size(tmp,1),
    for k=1:2,
      for l=(k+1):3,
        tmq(tmp(j,k),tmp(j,l))=tmq(tmp(j,k),tmp(j,l))+1;
        tmq(tmp(j,l),tmp(j,k))=tmq(tmp(j,l),tmp(j,k))+1;
      end
    end
  end
  [tma,tmb,tmc]=find(tmq); tmq=sparse(vn,vn);
  for j=1:size(tma,1),
    if(~(tmc(j)-1))
      tmq(b{i,3}(tma(j)),b{i,3}(tmb(j)))=1; tmq(b{i,3}(tmb(j)),b{i,3}(tma(j)))=1;
    end
  end
  tma=[b{i,3}(1)]; tmb=b{i,3}(1); [nth,tmc]=find(tmq(tmb,:));
  tmd=tmc(1); tma=[tma,tmd]; tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  while(tmb-b{i,3}(1))
      [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1); tma=[tma,tmd];
    tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  end
  b{i,6}=tma; b{i,5}=b{i,6}(1,1:b{i,2}); b{i,7}=2;
end
for i=1:bn,
  bcc(b{i,1}(1),b{i,1}(2))=i; bcc(b{i,1}(2),b{i,1}(1))=i;
end
bc=[];
for i=1:cn,
  bc{i,2}=[];
end
```

```
  for i=1:bn,
    bc{b{i,1}(1),2}=[bc{b{i,1}(1),2},i]; bc{b{i,1}(2),2}=[bc{b{i,1}(2),2},i];
  end
  n=bn;
  for i=1:bdrn,
    n=n+1; b{n,1}=bdr{i,1}; b{n,2}=bdr{i,2}; b{n,3}=bdr{i,3};
    tmp=ones(1,bdr{i,2}); b{n,4}=sparse(tmp,bdr{i,3},tmp,1,vn); tmp=[];
    for j=1:b{n,2},
      tmp=[tmp;v(b{n,3}(j),:)];
    end
    tma=max(tmp,[],1)-min(tmp,[],1); tmb=sortrows([1,2,3;tma]',2);
    tmb=round(tmb(:,1)); tmq=[tmp(:,tmb(1,1)),tmp(:,tmb(2,1))];
    tmp=delaunay(tmq(:,1),tmq(:,2)); tmq=sparse(b{n,2},b{n,2});
    for j=1:size(tmp,1),
      for k=1:2,
        for l=(k+1):3,
          tmq(tmp(j,k),tmp(j,l))=tmq(tmp(j,k),tmp(j,l))+1;
          tmq(tmp(j,l),tmp(j,k))=tmq(tmp(j,l),tmp(j,k))+1;
        end
      end
    end
    [tma,tmb,tmc]=find(tmq); tmq=sparse(vn,vn);
    for j=1:size(tma,1),
      if(~(tmc(j)-1))
        tmq(b{n,3}(tma(j)),b{n,3}(tmb(j)))=1; tmq(b{n,3}(tmb(j)),b{n,3}(tma(j)))=1;
      end
    end
    tma=[b{n,3}(1)]; tmb=b{n,3}(1); [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1);
    tma=[tma,tmd]; tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
    while(tmb-b{n,3}(1))
      [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1); tma=[tma,tmd];
      tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
    end
    b{n,6}=tma; b{n,5}=b{n,6}(1,1:b{n,2}); b{n,7}=1;
  end
  for i=(bn+1):n,
    bc{b{i,1},2}=[bc{b{i,1},2},i];
  end
  bn=size(b,1);
  for i=1:cn,
    bc{i,1}=size(bc{i,2},2);
  end
  fc=[];
  for i=1:cn,
    fc{i,2}=[];
  end
  for i=1:bn,
    for j=1:b{i,7},
      fc{b{i,1}(j),2}=[fc{b{i,1}(j),2},i];
    end
  end
  for i=1:cn,
    fc{i,1}=size(fc{i,2},2);
  end
  tma=[]; tmb=[];
  for i=1:cn,
    tmp=[]; tmf=[]; tmg=[];
    for j=1:vc{i,1},
      tmp=[tmp;v(vc{i,2},:)];
    end
    tmd=delaunayn(tmp);
    tmn=size(tmd,1);
    for j=1:tmn,
      tmq=[tmp(tmd(j,1),:);tmp(tmd(j,2),:);tmp(tmd(j,3),:);tmp(tmd(j,4),:)];
      tma=sum(tmq)/4; tmb=abs(det([tmq,ones(4,1)])/6); tmf=[tmf;tma*tmb]; tmg=[tmg,tmb];
    end
    vc{i,5}=sum(tmg); vc{i,4}=sum(tmf)/vc{i,5};
  end
  bb=[];
  for i=1:bn,
    if(~(b{i,7}-1))
      bb=[bb,i];
    end
  end
  vs{z,1}=c; vs{z,2}=v; vs{z,3}=vc; vs{z,4}=b; vs{z,5}=bc; vs{z,6}=fc; vs{z,7}=bb; g=[];
  for i=1:cn,
    g=[g;vc{i,4}];
  end
  gn=size(g,1); ca=g; can=gn;
end
mag=400;   % start statistics
for z=1:3,
  vc=vs{z,3}; cn=size(vc,1); tmp=[];
  for i=1:cn,
    tmp=[tmp;vc{i,5}];
  end
  tmp=mag*tmp; nc(z)=cn; sm(z)=mean(tmp); ss(z)=std(tmp);
  mt(z)=moment(tmp,3); mf(z)=moment(tmp,4);
end
mag=10; mag=mag^3; tmq=[];
for z=1:3,
  tmq{z}=[];
end
for z=1:3,
  for i=1:size(vs{z,1},1),
    tmq{z}=[tmq{z};vs{z,3}{i,5}];
  end
  tmq{z}=mag*tmq{z};
end
tmp=[];
for z=1:3,
  tmp=[tmp;tmq{z}];
end
tma=min(tmp); tmb=max(tmp); n=20; tmc=(tmb-tma)/n; tma=tma:tmc:tmb;
for i=1:z,
  tmb=hist(tmq{i},tma); figure(i); clf; axes(’FontSize’,13);
  bar(tma,tmb); xlabel(’volume’,’FontSize’,15); ylabel(’number of cells’,’FontSize’,15);
end
for zi=1:z,
  figure(z+zi); clf; hold on; v=vs{zi,2}; b=vs{zi,4}; bb=vs{zi,7}; bbn=size(bb,2);
  for i=1:bbn,
    tmp=[];
    for j=1:b{bb(i),2},
      tmp=[tmp;v(b{bb(i),5}(j),:)];
    end
    tmp=[tmp;tmp(1,:)]; fill3(tmp(:,1),tmp(:,2),tmp(:,3),b{bb(i),1});
  end
  axis equal; axis off; rotate3d; view(96,0);
end
for zi=1:z,
  figure(z+zi); clf; hold on; v=vs{zi,2}; b=vs{zi,4}; bb=vs{zi,7}; bbn=size(bb,2);
  for i=1:bbn,
    tmp=[];
    for j=1:b{bb(i),2},
      tmp=[tmp;v(b{bb(i),3}(j),:)];
    end
    plot3(tmp(:,1),tmp(:,2),tmp(:,3));
  end
  axis equal; axis off; rotate3d;
end
```

§ A.22 Defining irregular objects

```
% tioa.m, three irregular object algorithms, Kit Tiyapan (c) 12th November 2002
% tioa1.m
clear all;
v{1,1}=[1,1;3,4;3,8;3.5,9;4,8;4,4;6,1;4,3;4,2;3.5,1;3,2;3,3];
v{2,1}=[1,2;1,8;1.5,9;2,8;2,2;1.5,1]; v{3,1}=[1,7;3.5,9;6,7;4,8;4,2;3.5,1;3,2;3,8];
vn=size(v,1); mag=10;
for i=1:vn,
  v{i,2}=size(v{i,1},1);
end
for i=1:vn,
  tmp=max(v{i,1}); v{i,3}=sparse(tmp(1),tmp(2));
  for j=1:v{i,2},
    tmp=mag*[v{i,1};v{i,1}(1,:)]; tmx=tmp((j+1),1)-tmp(j,1);
    tmy=tmp((j+1),2)-tmp(j,2); tms=sign(tmx);
    if(~tmx)
      for k=tmp(j,2):sign(tmy):tmp((j+1),2),
        v{i,3}(tmp(j,1),k)=1;
      end
    else
      if(~tmy)
        for k=tmp(j,1):tms:tmp((j+1),1),
          v{i,3}(k,tmp(j,2))=1;
        end
      else
        tmm=tmy/tmx; tmc=tmp(j,2)-tmm*tmp(j,1);
        for k=tmp(j,1):tms:tmp((j+1),1),
          tmb=round(tmm*k+tmc); v{i,3}(k,tmb)=1;
        end
      end
    end
  end
end
tma=0;
ost(1,:)=[0,0];
for i=1:(vn-1),
  tmp=max(v{i}); tma=tma+tmp(1); ost((i+1),:)=[tma,0];
end
tmx=[]; tmy=[];
for i=1:vn,
  [tma,tmb]=find(v{i,3}); tmn=size(tma,1); tma=mag*ost(i,1)*ones(tmn,1)+tma;
  tmb=mag*ost(i,2)*ones(tmn,1)+tmb; tmx=[tmx;tma]; tmy=[tmy;tmb];
end
tmn=size(tmx,1); clf;
for i=1:tmn,
  tma=tmx(i)-.5; tmb=tmx(i)+.5; tmc=tmy(i)-.5; tmd=tmy(i)+.5;
  fill([tma,tmb,tmb,tma],[tmc,tmc,tmd,tmd],k); hold on;
end
axis equal; axis off;
% tioa2.m
clear all; v{1,2}=[1,1;3,4;3,8;3.5,9;4,8;4,4;6,1;4,3;4,2;3.5,1;3,2;3,3];
v{2,2}=[1,2;1,8;1.5,9;2,8;2,2;1.5,1]; v{3,2}=[1,7;3.5,9;6,7;4,8;4,2;3.5,1;3,2;3,8];
vn=size(v,1); mag=10; iny=1000; tmn=0;
for i=1:vn,
  v{i,1}=size(v{i,2},1); tmm=max(max(v{i,2}));
  if(tmm>tmn)
    tmn=tmm;
  end
end
for i=1:vn,
  tmp=[v{i,2};v{i,2}(1,:)];
  for j=1:v{i,1},
    tma=tmp((j+1),1)-tmp(j,1); tmb=tmp((j+1),2)-tmp(j,2); tmc=max(abs(tma),abs(tmb));
    if(tmc<tmn)
      tmn=tmc;
    end
    if(tma)
      v{i,3}(j,1)=tmb/tma;
    else
      v{i,3}(j,1)=iny;
    end
  end
end
stp=tmn/mag;
for i=1:vn,
  tmp=round(max(v{i,2})/stp); v{i,4}=sparse(tmp(1),tmp(2)); tmp=[v{i,2};v{i,2}(1,:)];
  for j=1:v{i,1},
    if(abs(v{i,3}(j))<0.5)
      tma=round(tmp(j,1)/stp); tmb=round(tmp((j+1),1)/stp);
      for k=tma:sign(tmb-tma):tmb,
        tmc=round(tmp(j,2)/stp+(k-tma)*v{i,3}(j)); v{i,4}(k,tmc)=1;
      end
    else
      tma=round(tmp(j,2)/stp); tmb=round(tmp((j+1),2)/stp);
      for k=tma:sign(tmb-tma):tmb,
        tmc=round(tmp(j,1)/stp+(k-tma)/v{i,3}(j)); v{i,4}(tmc,k)=1;
      end
    end
  end
end
for i=1:vn,
  v{i,5}=max(v{i,2});
end
tmp=[]; tmq=[]; tmr=[]; oft=0;
for i=1:vn,
  [tma,tmb,tmc]=find(v{i,4}); tma=(oft/stp)*ones(size(tma))+tma;
  tmp=[tmp;tma]; tmq=[tmq;tmb]; tmr=[tmr;tmc]; oft=oft+v{i,5}(1);
end
tmn=size(tmp,1); clf;
for i=1:tmn,
  tma=tmp(i)-.5; tmb=tmp(i)+.5; tmc=tmq(i)-.5; tmd=tmq(i)+.5;
  fill([tma,tmb,tmb,tma],[tmc,tmc,tmd,tmd],k); hold on;
 end
axis equal; axis off;
% tiao3.m
clear all; v{1,2}=[1,1;3,4;3,8;3.5,9;4,8;4,4;6,1;4,3;4,2;3.5,1;3,2;3,3];
v{2,2}=[1,2;1,8;1.5,9;2,8;2,2;1.5,1]; v{3,2}=[1,7;3.5,9;6,7;4,8;4,2;3.5,1;3,2;3,8];
vn=size(v,1); tma=[];
for i=1:vn,
  v{i,1}=size(v{i,2},1); tmp=[v{i,2};v{i,2}(1,:)];
  for j=1:v{i,1},
    tma=diff(tmp);
  end
end
tmb=max(min(abs(tma))); mag=10; mag=mag/tmb;
for i=1:vn,
  tmc=min(v{i,2});
  v{i,4}=ceil(mag*v{i,2}-[tmc(1)*ones(v{i,1},1),tmc(2)*ones(v{i,1},1)]);
  v{i,3}=max(v{i,4});
end
for i=1:vn,
  tmz=[v{i,4};v{i,4}(1,:)]; tma=diff(tmz); tmm=tma(:,2)./tma(:,1); tmp=[]; tmq=[];
  for j=1:v{i,1},
    tms=sign(tmz((j+1),1)-tmz(j,1)); tmx=tmz(j,1):tms:tmz((j+1),1);
    tmu=tmz(j,1)*ones(size(tmx)); tmv=tmz(j,2)*ones(size(tmx));
    tmy=ceil(tmm(j)*(tmx-tmu)+tmv); tmn=size(tmy,2); tma=diff(tmy); tmb=[]; tmd=[];
    for k=1:(tmn-1),
      tmb=[tmb,tmx(k)]; tmc=min(tmy(k),v{i,3}(2)); tmd=[tmd,tmc];
      for l=1:abs(tma(k)),
        tmd=[tmd,min(v{i,3}(2),(tmc+l*tms))]; tmb=[tmb,tmx(k)];
      end
    end
    tmx=tmb; tmy=tmd; tmp=[tmp,tmx];tmq=[tmq,tmy];
  end
  tma=ones(size(tmp)); v{i,5}=sparse(tmp,tmq,tma,v{i,3}(1),v{i,3}(2));
end
for i=1:vn,
  tmz=[v{i,4};v{i,4}(1,:)]; tma=diff(tmz); tmm=tma(:,1)./tma(:,2); tmp=[]; tmq=[];
  for j=1:v{i,1},
    tms=sign(tmz((j+1),2)-tmz(j,2)); tmy=tmz(j,2):tms:tmz((j+1),2);
    tmv=tmz(j,2)*ones(size(tmy)); tmu=tmz(j,1)*ones(size(tmy));
    tmx=ceil(tmm(j)*(tmy-tmv)+tmu); tmn=size(tmx,2); tma=diff(tmx); tmb=[]; tmd=[];
    for k=1:(tmn-1),
      tmb=[tmb,tmy(k)]; tmc=min(tmx(k),v{i,3}(1)); tmd=[tmd,tmc];
      for l=1:abs(tma(k)),
        tmd=[tmd,min(v{i,3}(1),(tmc+l*tms))]; tmb=[tmb,tmy(k)];
      end
    end
    tmy=tmb; tmx=tmd; tmp=[tmp,tmx]; tmq=[tmq,tmy];
  end
  tma=ones(size(tmp)); v{i,5}=v{i,5}|sparse(tmp,tmq,tma,v{i,3}(1),v{i,3}(2));
end
tmp=[]; tmq=[]; osx=0; osy=0; tma=0; tmb=0;
for i=2:vn,
  tma=tma+v{(i-1),3}(1); tmb=0; osx=[osx,tma]; osy=[osy,tmb];
end
for i=1:vn,
  [tma,tmb]=find(v{i,5}); tmc=ones(size(tma));
  tmp=[tmp;(tma+tmc*osx(i))]; tmq=[tmq;(tmb+tmc*osy(i))];
end
tmn=size(tmp,1); clf;
for i=1:tmn,
  tma=tmp(i)-.5; tmb=tmp(i)+.5; tmc=tmq(i)-.5; tmd=tmq(i)+.5;
  fill([tma,tmb,tmb,tma],[tmc,tmc,tmd,tmd],k); hold on;
end
axis equal; axis off;
```

§ A.23 Investigating the distribution observed from within the network

```
% dstnvst.m; Kit Tiyapan, 17th November 2002
% (x,y) in square domain
clear all; Dim=2; St=sum(100*clock); rand('state',St); N=1000;
X=rand(N,Dim); Ta=delaunay(X(:,1),X(:,2)); TaN=size(Ta,1); EXX=sparse(N,N);
E=[]; DL=[]; xc=1/2; yc=1/2; LB=0.05; UB=1-LB; Xin=ones(N,1);
for i=1:N,
  if((min(X(i,1),X(i,2))<LB) | (max(X(i,1),X(i,2))>UB))
    Xin(i,1)=0;
  end
end
% or (x,y) in circular domain
clear all; Dim=2; St=sum(100*clock); rand('state',St); N=1000;
TwoPi=2*pi; X=[]; R=[]; xc=1/2; yc=1/2;
for i=1:N,
  Tmp=rand(1,2); TmpB=(Tmp(1)-xc); TmpC=(Tmp(2)-yc); TmpA=(TmpB*TmpB+TmpC*TmpC);
  while(TmpA>(1/4))
    Tmp=rand(1,2); TmpB=(Tmp(1)-xc); TmpC=(Tmp(2)-yc); TmpA=(TmpB*TmpB+TmpC*TmpC);
  end
  X=[X;Tmp]; R=[R;TmpA];
end
Ta=delaunay(X(:,1),X(:,2));
% or (r,theta) in a circle
clear all; Dim=2; St=sum(100*clock); rand('state',St); N=1000; TwoPi=2*pi;
R=rand(N,1); Th=TwoPi*rand(N,1); X=[R.*cos(Th), R.*sin(Th)]; Ta=delaunayn(X);
% and then
TaN=size(Ta,1); EXX=sparse(N,N); E=[]; DL=[]; xc=0; yc=0; B=0.95; Xin=ones(N,1);
for i=1:N,
  if(R(i,1)>B)
    Xin(i,1)=0;
  end
end
% then
T=[];
for i=1:TaN,
  include=1;
  for j=1:(Dim+1),
    if(~Xin(Ta(i,j)))
      include=0; break;
    end
  end
  if(include)
    T=[T;Ta(i,:)];
  end
end
TN=size(T,1);
for i=1:TN,
  Tmp=[T(i,:),T(i,1)];
  for j=1:(Dim+1),
    if(~EXX(Tmp(j),Tmp(j+1)))
      E=[E;[Tmp(j),Tmp(j+1)]]; dx=X(Tmp(j+1),1)-X(Tmp(j),1);
      dy=X(Tmp(j+1),2)-X(Tmp(j),2); length=sqrt(dx*dx + dy*dy);
      xm=X(Tmp(j),1)+(dx/2); ym=X(Tmp(j),2)+(dy/2); dx=xm-xc; dy=ym-yc;
      dist=sqrt(dx*dx + dy*dy); DL=[DL;[dist,length]];
      EXX(Tmp(j),Tmp(j+1))=1; EXX(Tmp(j+1),Tmp(j))=1;
    end
  end
end
EN=size(E,1);
% plot
figure(1); clf; hold on;
for i=1:EN,
  plot([X(E(i,1),1),X(E(i,2),1)],[X(E(i,1),2),X(E(i,2),2)]);
end
axis equal; axis off;
n=20; int=1/n; Tmp=zeros(n,2);
for i=1:EN,
  TmpA=ceil(DL(i,1)/int); Tmp(TmpA,1)=Tmp(TmpA,1)+DL(i,2); Tmp(TmpA,2)=Tmp(TmpA,2)+1;
end
TmpA=[];
for i=1:n,
  if(Tmp(i,2))
    TmpA=[TmpA;Tmp(i,:)];
  end
```

```
end
Ld=TmpA(:,1)./TmpA(:,2); LdN=size(Ld,1); Tmp=(int/2):int:(LdN*int); Ld=[Tmp',Ld];
figure(2); clf; bar(Ld(:,1),Ld(:,2),'m');
xlabel('Distance from centre','FontSize',13); ylabel('Average length','FontSize',13);
```

§ A.24 Miscellaneous functions

```
% findfarea.m, finds face area, (c) Kit Tiyapan, February, 2001.
function [fca, ppr] = findfarea(odvc)
nvthf=size(odvc,1); stp=floor(nvthf/3); ndpt =1+stp; rdpt =1+2*stp;
nmvc= cross((odvc(ndpt,:)-odvc(1,:)),(odvc(rdpt,:)-odvc(1,:)));
clov =[odvc; odvc(1,:)]; xsq =nmvc(1,1)*nmvc(1,1); ysq =nmvc(1,2)*nmvc(1,2);
zsq =nmvc(1,3)*nmvc(1,3); znmvc =sqrt(xsq+ysq+zsq); nmnmv =nmvc/znmvc; socp =0;
for i=1:nvthf,
  socp =socp+cross(clov(i,:),clov((i+1),:));
end
fca =(abs(dot(nmnmv, socp)))/2;
nvct =ones(3,1); crdm =[odvc(1,:); odvc(ndpt,:); odvc(rdpt,:)];
aprm =det([nvct, crdm(:,2), crdm(:,3)]); bprm =det([crdm(:,1),  nvct, crdm(:,3)]);
cprm =det([crdm(:,1), crdm(:,2), nvct]);
dprm =det([crdm(:,1), crdm(:,2), crdm(:,3)]); ppr =[aprm, bprm, cprm, dprm];
% ordervertices.m, orders the vertices in a list, (c) Kit Tiyapan, February 2001.
function [odvc] = ordervertices(vnsp)
zvnsp =size(vnsp,1);
if (zvnsp==3)
  odvc =vnsp; return;
end
vmny =1; vmxy =1;
for i=2:zvnsp,
  if (vnsp(i,2) < vnsp(vmny,2))
    vmny =i;
  elseif (vnsp(i,2) > vnsp(vmxy,2))
    vmxy =i;
  end
end
ndps =[vnsp(vmny,1:2); vnsp(vmxy,1:2)]; aprm =det([[1;1],ndps(:,2)]);
bprm =det([ndps(:,1),[1;1]]); cprm =det(ndps); vrts =vnsp(vmny,:); vlfs =vnsp(vmxy,:);
for i=1:zvnsp,
  if ((i~=vmny) & (i~=vmxy))
    lfpq =aprm*vnsp(i,1)+bprm*vnsp(i,2)+cprm;
    if (lfpq > 0)
      vrts =[vrts; vnsp(i,:)];
    elseif (lfpq < 0)
      vlfs =[vlfs; vnsp(i,:)];
    end
  end
end
zvrts =size(vrts,1); zvlfs =size(vlfs,1);
if(zvrts >= 3)
  if (vrts(1,2)==vrts(2,2))
    if (vrts(1,1) > vrts(2,1))
      vrts =[[vrts(2,:); vrts(1,:)]; vrts(1,3:zvrts)];
    end
  end
  Angle =[];
  for i=2:zvrts,
    xcd =vrts(i,1)-vrts(1,1); ycd =vrts(i,2)-vrts(1,2); ang =[ang; atan(ycd/xcd)];
  end
  vrts =sort([ang,vrts(2:zvrts,:)],1); svrts =[vrts(1,:); vrts(:,2:5)];
else
  svrts =vrts;
end
if(zvlfs >= 3)
  if (vlfs(1,2)==vlfs(2,2))
    if (vlfs(1,1) < vlfs(2,1))
      vlfs =[vlfs(2,:); vlfs(1,:)];
    end
  end
  ang =[];
  for i=2:zvlfs,
    xcd =vlfs(i,1)-vlfs(1,1); ycd =vlfs(i,2)-vlfs(1,2); ang =[ang; atan(ycd/xcd)];
  end
  vlfs =sort([ang,vlfs(2:zvlfs,:)],1); svlfs =[vlfs(1,:); vlfs(:,2:5)];
else
  svlfs =vlfs;
end
odvc =[svrts; svlfs];
% perc.m, function to find percolation, Kit Tiyapan, (c) 21st November, 2002
function [Pc,Cord,TSeries] = perc(N,LMat,UMat,NeMat)
Blocked=randperm(N); NClusA=0; Perco=0;
```

```
for i=1:N,
 Joined=0;
 for j=1:NClusA,
  if(ClusA{j,3}(1,Blocked(1,i))~=0)
   ClusA{j,1}=ClusA{j,1}+1; ClusA{j,2}(1,Blocked(1,i))=1;
   ClusA{j,3}=ClusA{j,3} | NeMat(Blocked(1,i),:); Joined=1;
  end
  if(Joined==1)
   for k=1:4,
    ClusB{1,k}=ClusA{j,k};
   end
   NClusB=1;
   if(j==1)
    Tmp=ClusA; clear ClusA;
    for k=1:(NClusA-1),
     for l=1:4,
      ClusA{k,l}=Tmp{(k+1),l};
     end
    end
   elseif(j==NClusA)
    Tmp=ClusA; clear ClusA;
    for k=1:(NClusA-1),
     for l=1:4,
      ClusA{k,l}=Tmp{k,l};
     end
    end
   else
    Tmp=ClusA; clear ClusA;
    for k=1:(j-1),
     for l=1:4,
      ClusA{k,l}=Tmp{k,l};
     end
    end
    for k=j:(NClusA-1),
     for l=1:4,
      ClusA{k,l}=Tmp{(k+1),l};
     end
    end
   end
   for k=1:(NClusA-1),
    if(sum(ClusA{k,2} & ClusB{1,3}) ~= 0)
     ClusB{1,1}=ClusB{1,1}+ClusA{k,1}; ClusB{1,2}=ClusB{1,2} | ClusA{k,2};
     ClusB{1,3}=ClusB{1,3} | ClusA{k,3}; ClusB{1,4}=ClusB{1,4} | ClusA{k,4};
    else
     NClusB=NClusB+1;
     for l=1:4,
      ClusB{NClusB,l}=ClusA{k,l};
     end
    end
   end
   if((sum(full(LMat & ClusB{1,2}))~=0) & (sum(full(UMat & ClusB{1,2}))~=0))
    ClusB{1,4}=1; Perco=1;
   end
   NClusA=NClusB; ClusA=ClusB; clear ClusB; break;
  end
 end
 if(Joined==0)
  NClusA=NClusA+1; ClusA{NClusA,1}=1; ClusA{NClusA,2}=sparse(1,Blocked(1,i),1,1,N);
  ClusA{NClusA,3}=NeMat(Blocked(1,i),:); ClusA{NClusA,4}=0;
 end
 TSeries{1}{i,1}=ClusA; TSeries{1}{i,2}=Perco;
end
Tmp=Blocked; Blocked=[];
for i=1:N,
 Blocked=[Blocked,Tmp(1,(N-i+1))];
end
Nc=0; TSnap=[];
for i=1:N,
 if(TSeries{1}{i,2})
  Nc=i; break;
 end
end
Pc=Nc/N; Cord=mean(sum(NeMat,2)); clear ClusA ClusB; NClusA=0; Perco=0;
for i=1:N,
 Joined=0;
 for j=1:NClusA,
  if(ClusA{j,3}(1,Blocked(1,i))~=0)
   ClusA{j,1}=ClusA{j,1}+1; ClusA{j,2}(1,Blocked(1,i))=1;
   ClusA{j,3}=ClusA{j,3} | NeMat(Blocked(1,i),:); Joined=1;
  end
  if(Joined==1)
   for k=1:4,
        ClusB{1,k}=ClusA{j,k};
      end
      NClusB=1;
      if(j==1)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      elseif(j==NClusA)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
      else
        Tmp=ClusA; clear ClusA;
        for k=1:(j-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
        for k=j:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      end
      for k=1:(NClusA-1),
        if(sum(ClusA{k,2} & ClusB{1,3}) ~= 0)
          ClusB{1,1}=ClusB{1,1}+ClusA{k,1}; ClusB{1,2}=ClusB{1,2} | ClusA{k,2};
          ClusB{1,3}=ClusB{1,3} | ClusA{k,3}; ClusB{1,4}=ClusB{1,4} | ClusA{k,4};
        else
          NClusB=NClusB+1;
          for l=1:4,
            ClusB{NClusB,l}=ClusA{k,l};
          end
        end
      end
      if((sum(full(LMat & ClusB{1,2}))~=0) & (sum(full(UMat & ClusB{1,2}))~=0))
        ClusB{1,4}=1; Perco=1;
      end
      NClusA=NClusB; ClusA=ClusB; clear ClusB; break;
    end
  end
  if(Joined==0)
    NClusA=NClusA+1; ClusA{NClusA,1}=1; ClusA{NClusA,2}=sparse(1,Blocked(1,i),1,1,N);
    ClusA{NClusA,3}=NeMat(Blocked(1,i),:); ClusA{NClusA,4}=0;
  end
  TSeries{2}{i,1}=ClusA; TSeries{2}{i,2}=Perco;
end
Nc=0; TSnap=[];
for i=1:N,
  if(TSeries{2}{i,2})
    Nc=i; break;
  end
end
Pc=[Pc,Nc/N]; Cord=[Cord,mean(sum(NeMat,2))];
```

§ A.25 Percolated traffic networks

```
% Amsterdam
clear all; twn='Amsterdam'; T=7; B=0; L=0; R=7;
V=[2.5,3.8; 2.6,3.9; 2.6,4.3; 2.5,4.2; 2.5,4.5; 2.7,4.6; 2.9,4.7; 2.8,4.3; 2.8,3.8; 2.9,3.2;
2.8,3.1; 2.5,2.9; 2.4,2.8; 2.1,3.2; 2,3.3; 2,3.8; 2.1,4.5; 2.1,5.4; 2.2,5.9; 2.3,6.5;
2.4,6.6; 2.5,6.5; 2.5,T; 2.7,6.9; 2.9,6.8; 2.6,6.5; 2.5,6.2; 2.3,6.2; 2.3,6; 2.5,5.9;
2.4,5.4; 2.2,5.4; 2.2,5.1; 2.4,5.1; 2.4,4.9; 2.2,4.9; 2.2,4.5; 2.4,4.5; 2.3,3.8; 2.2,3.8;
2.2,3.6; 2.2,3.4; 2.4,3.5; 2.6,3.6; 2.7,4.8; 2.5,4.7; 2.8,5.1; 3.1,5.1; 3.1,5.4; 2.7,5.3;
2.6,5.4; 2.8,5.5; 3.1,5.6; 2.8,5.8; 2.7,5.8; 2.8,6.5; 3,6.5; 3.1,5.9; 3.3,6.3; 3.4,6.6;
3.2,6.6; 3.1,6.7; 3.7,T; 3.5,T; 3.3,T; 3,T; 2.7,T; 2,T; 1.8,T; 1.5,T;
1.4,T; 1.1,6.8; 1,6.7; .9,6.9; .9,T; .4,6.9; .3,T; .4,6.7; .5,6.5; .5,6.4;
.6,6.3; .7,6; .3,5.8; L,5.6; L,6; L,6.3; L,6.4; L,6.6; L,6.7; L,6.9;
.9,6.1; .9,5.8; .4,5.5; .4,5.3; .9,5.5; .9,5.4; .9,5.3; .9,5.1; .6,4.9; .5,5.1;
.9,4.8; .7,4.7; .3,4.5; L,4.3; L,4.7; L,5.1; L,5.3; .9,4.5; .9,4.2; .5,3.9;
.3,3.7; .3,3.5; .6,3.7; .8,3.8; .8,3.4; .6,3.6; .5,3.5; .5,3.3; .8,3.4; .8,3.2;
.6,3.1; .4,2.9; .1,3.1; .3,2.6; .8,2.7; .9,2.6; .5,2.4; .3,2.2; L,2.1; .7,1.3;
.4,1; L,1.4; L,.9; .5,.7; L,.6; .6,.6; L,.3; .9,.5; 1.5,.4; 1.6,.3;
1.1,.3; 1.3,B; .6,B; 1.9,.1; 1.5,B; 2.3,B; 2.2,.3; 2.8,B; 2.5,.1; 2.6,.3;
2.3,.5; 2.2,.4; 1.4,1.2; 1.6,1.4; 1.8,1.5; 1.5,2; 1.6,2.2; 1.2,2.6; 1.1,2.7; 1,2.9;
1.5,2.8; 1.5,2.9; 1.6,3; 1.7,2.9; 1.4,3.1; 1.1,3.8; 1.5,3.8; 1.6,3.8; 1.6,4.5; 1.4,4.5;
1.1,4.5; 1.2,5.4; 1.5,5.4; 1.2,6.2; 1.5,6.2; 1.6,6.2; 1.3,6.6; 1.6,6.5; 1.4,6.9; 1.7,6.7;
1.8,6.9; 1.9,6.8; 1.9,6.7; 1.9,2.5; 2.1,2.5; 2.8,1.8; 2.7,1.6; 2.6,1.4; 2.4,1; 2.7,.8;
2.7,.7; 2.6,.5; 2.6,.4; 2.9,.2; 2.9,.3; 3.1,B; 3.3,.1; 3.3,.3; 3.4,.4; 3.4,.6;
3.5,.7; 3.5,.9; 3.5,1.3; 3.6,1.5; 3.6,1.7; 3.7,2.1; 3.8,2.3; 3.8,2.6; 2.7,3; 3.9,3.1;
4.1,3.3; 3.6,4.1; 3.9,4.1; 4,4.1; 4.1,4.1; 4.3,4.1; 3.7,4.4; 3.8,4.4; 4,4.3; 4.1,4.3;
4.2,4.5; 4.4,4.4; 3.7,4.6; 3.9,4.6; 3.7,4.7; 3.9,4.7; 4.2,4.7; 4.4,4.7; 4.4,4.6; 3.8,5;
3.9,5.1; 3.9,5; 4.3,5; 4.3,5.1; 3.8,5.3; 4,5.3; 4.2,5.3; 4.4,5.2; 3.9,5.6; 4.5,5.3;
4.6,5.3; 4.7,5.2; 4.5,5; 4.6,5; 4.5,4.5; 4.6,4.4; 4.7,4.6; 4.7,4.7; 4.8,4.5; 5.1,4.5;
5,4.3; 4.9,4.3; 4.8,4.1; 4.4,4.1; 4.2,3.6; 4.3,3.5; 4.5,3.7; 4.7,3.6; 4.6,3.3; 4.5,3;
5.1,3.6; 5.4,4.3; 5.2,4.4; 5.2,4.5; 5.4,4.6; 5.4,4.4; 5.5,4.5; 5.7,4.4; 5.7,4.3; 5.8,4.2;
5.7,4.1; 5.6,4.2; 5.2,3.5; 5.3,3.5; 5.6,3.4; 6.1,3; 5.9,4; 6,4.2; 6.3,4; 6.2,3.7;
6.5,3.4; 6.7,3.7; 6.6,3.5; 6.4,4.2; 6.5,4.3; 6.9,4.6; 6.7,4.8; R,5; 6.3,4.6; 6.5,4.9;
6.7,5.1; 6.9,5.2; 6,4.3; 5.9,4.7; 6,4.6; 6.1,4.7; 6.2,4.8; 5.8,5.3; 6,5.3; 6,5.2;
6.2,5.3; 5.9,5.5; 6.1,5.4; 5.8,5.3; 5.6,5.4; 5.5,5.4; 5.4,5.4; 5.4,5.2; 5.1,5.3; 5.2,5.5;
5.3,5.6; 5.4,5.8; 5.4,5.9; 5.6,5.7; 5.7,5.8; 5.9,5.8; 5.7,6; 5.5,6.1; 5.4,6.2; 5.3,6.2;
5.2,6.3; 5.1,6.1; 5,6; 4.8,5.8; 4.5,5.8; 4.7,5.7; 4.9,5.6; 5,5.5; 4.5,5.4; 4.1,5.6;
3.9,5.9; 3.2,6.1; 4.3,6.3; 4.4,6.3; 4.3,5.9; 4.6,6.3; 4.6,6.5; 4.8,6.4; 5,6.6; 5.4,6.5;
5.6,6.9; 5.1,6.8; 4.7,T; 5.3,6.8; 5.5,6.7; 5.6,6.9; 5.7,6.8; 5.8,6.7; 5.9,6.7; 6.1,6.7;
5.5,6.6; 5.7,6.5; 5.8,6.5; 5.9,6.4; 6.2,6.6; 6.4,6.5; 6.2,6.4; 6.4,6.3; 6.2,6.3; 6.5,6.1;
6.7,6.3; 6.7,6.7; 6.5,T; 6.9,6.5; 6.1,6.1; 6.1,6; 6.4,5.8; 6.6,5.6; 6.7,5.7; 6.8,5.8;
6.1,5.6; 6.2,5.7; 6.3,5.5; 6.4,5.4; 6.4,5.6; 6.5,5.5; 6.3,5.2; 6.4,5.1; 6.5,5; 6.6,5.3;
6.5,5.4; 6.7,5.4; 5.3,3; 5.5,2.9; 5.2,2.6; 5.5,2.6; 5.8,2.6; 5.8,2.5; 5.9,2.4; 5.2,2.2;
5.1,2.3; 4.5,2.2; 4.6,2; 4.4,2; 4.5,1.5; 4.5,1.3; 4.5,1; 4.5,.7; 4.1,.6; 4.1,.4;
4,.2; 4.6,.6; 4.6,.3; 4.6,.1; 4.7,.1; 4.7,.4; 4.7,.5; 4.7,.6; 4.6,1; 4.6,1.4;
4.6,1.5; 5.3,1.6; 5.3,1.6; 5.4,1.2; 5.2,1.2; 5.3,.8; 5.5,.9; 5.5,.7; 5.5,.5; 5.5,.4;
5.2,.2; 5.2,.1; 5.6,.1; 6,.5; 6.1,.4; 6.2,.2; 6.5,.3; 6.5,.5; 6.4,.6; 6.4,.6;
6.3,1; 6.3,1.2; 6.2,1.5; 6.1,1.7; 6,2; 6.3,2.6; 6.3,2.5; 6.4,2.1; 6.5,1.9; 6.5,1.6;
6.6,1.4; 6.7,1.3; R,2.9; R,2.7; T,2.4; R,2.3; R,1.8; R,1.6; R,1.4; R,.5;
2.5,1.1; 1.1,6.6; 1,6.4; L,4.8; L,3.6; L,3.3; L,3.1; L,.1; 1.6,5.4; 4.6,B;
4.3,5.5; 4.5,4.3; 5.9,4.4; 6.9,4; R,4.3; 6.1,5; 5,6.3; 4.9,6.2; 4.8,6.1; 4.9,6.8;
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EN=size(E,1);
Tmp=[11,13,1,3; 11,13,1.2,2.9; 20,23,.6,3.1; 34,51,2.6,2.4; 34,53,3,2.6; 42,43,1.5,1.5;
45,221,2.7,1.3; 45,221,2.6,1.2; 45,221,2.5,1; 45,221,2.4,.7; 56,58,4.3,2.7; 63,65,4.8,3.6;
65,66,4.6,4; 72,78,2.7,3.3; 129,132,3.7,5.6; 130,131,4.1,5.3; 130,149,4.6,5.4;
136,143,3.3,4.9; 136,143,3.3,4.8; 138,139,2.7,4.7; 152,164,6.5,5.4; 154,156,5.3,4.7;
176,255,5.7,3; 179,180,6.4,3.2; 200,201,3.6,1.4; 201,267,4.1,.5; 221,226,2.9,.9;
226,227,3.3,.5];
lmp=20/(3*2.54); lrl=1000; vx=[L,B;R,B;R,T;L,T];
c{1,2}=[281,233,216,205,208,269,284,(VN+1)]; c{2,2}=[285,227,221,45,216,233];
c{3,2}=[227,197,199,57,54,46,45,221,227]; c{4,2}=[286,266,197,227,285];
c{5,2}=[264,193,286,(VN+2)]; c{6,2}=[261,189,193,264];
c{7,2}=[286,193,189,60,199,197,266]; c{8,2}=[216,45,46,47,40,9,11,14,269,208,205];
c{9,2}=[269,14,21,215]; c{10,2}=[14,11,83,110,96,92,90,21];
c{11,2}=[46,54,53,70,76,83,11,9,40,47]; c{12,2}=[199,60,61,63,65,66,146,70,53,54,57];
c{13,2}=[189,187,166,152,61,60]; c{14,2}=[253,166,187,189,261];
c{15,2}=[253,166,152,287,(VN+3)]; c{16,2}=[61,152,287,151,146,66,65,63];
c{17,2}=[70,146,151,249,83,76]; c{18,2}=[83,249,248,110]; c{19,2}=[110,248,241,96];
c{20,2}=[96,241,(VN+4),239,92]; c{21,2}=[92,239,215,21,90]; cn=size(c,1);
% Freiburg
clear all; twn='Freiburg'; T=5; B=0; L=0; R=6;
V=[4.2,1.9; 3.5,2.1; 2.3,2.9; 2.4,3.3; 2.8,4.5; 3.3,T; 2.4,4.7; 2.4,T; 2.1,4.8; 2.2,T;
1.6,4.9; 1.2,T; 1.9,1.7; .9,4.3; .1,4.7; .3,T; L,4.8; L,4.4; .7,3.7; .3,3.2;
.7,3.1; 1.2,3; 1.4,2.5; 1.5,2.9; 1.6,3.6; 2,3.4; 1.9,2.9; 1.5,2.3; 1.7,2; 1.9,2;
2.3,2; 2.5,1.3; 3.4,1.2; 4.1,1.1; 4.8,1.2; 5.4,1.4; 5.7,1.5; R,1.5; R,1.8; 5.7,1.7;
5.4,1.9; 5.6,2.2; 4.9,1.9; R,3.4; 5.8,1.1; 4.7,.7; 4.1,.7; 3.3,.7; 4.7,.1; 5.3,B;
5.7,B; R,1.1; 4,.2; 3.3,.2; 3.2,B; 3.9,B; 2.8,.7; 2.6,B; 2,B; 2.1,.6;
2.7,.4; 2.1,.8; 1.5,.6; 1.3,.4; 1.1,.7; 1.2,1.3; .8,B; .4,1.5; L,1.6; L,1.4;
L,1.8; .2,2.2; .6,1.8; 2.2,1.3; L,2.4];
VN=size(V,1);
```

```
E=[1,2; 1,34; 1,43; 2,3; 2,33; 3,4; 3,27; 3,31; 4,5; 4,26; 5,6; 5,7; 7,8; 7,9; 9,10; 9,11;
11,12; 11,14; 12,14; 13,29; 13,74; 13,75; 14,15; 14,19; 15,16; 15,17; 15,18; 19,20; 19,22;
19,25; 20,21; 20,75; 21,22; 22,23; 22,24; 23,24; 23,28; 24,25; 24,27; 25,26; 26,27; 28,29;
28,30; 29,30; 30,31; 31,32; 32,33; 32,74; 33,34; 33,48; 34,35; 34,47; 35,36; 35,46; 35,43;
36,37; 36,41; 36,45; 37,40; 37,38; 37,45; 40,41; 40,42; 41,42; 41,43; 42,44; 45,46; 45,51;
45,52; 46,47; 46,49; 47,53; 47,48; 48,54; 48,57; 49,50; 49,53; 53,54; 53,56; 54,55; 57,61;
57,62; 58,61; 59,60; 60,61; 60,62; 62,66; 62,63; 62,74; 63,64; 63,65; 64,65; 64,67; 65,66;
66,68; 68,69; 68,70; 71,72; 71,73; 72,73; 72,75;
73,74];
EN=size(E,1);
Tmp=[11,14,1.3,4.2; 12,14,1,4.6; 13,75,1.9,1.6; 13,76,1.8,1.5; 19,20,.5,3.6;  28,30,1.8,2.4;
31,32,3.2,1.9; 34,35,4.5,1.1; 36,41,5.3,1.7; 64,65,1,.5; 73,74,2,1.2];
lmp=2.9; lrl=500; vx=[L,B;R,B;R,T;L,T];
c{1,2}=[67,74,71,(VN+1)]; c{2,2}=[55,33,74,67]; c{3,2}=[52,36,33,55,(VN+2)];
c{4,2}=[44,42,41,36,52]; c{5,2}=[36,41,42,4,74,33]; c{6,2}=[74,4,14,75,71];
c{7,2}=[75,14,17]; c{8,2}=[17,14,12,(VN+4)]; c{9,2}=[12,14,4,6]; c{10,2}=[6,4,42,44,(VN+3)];
cn=size(c,1);
% Manchester
clear all; twn='Mancheter'; T=9; B=0; L=0; R=9;
V=[.7,.8; 1.9,.8; 1.9,.1; 2.6,.8; 2.6,.3;3.5,.3; 3.8,.3; 3.8,1; 3.9,.3; 4.1,.3;
 4.3,.3; 4,1; 5,.3; 5.3,.6; 5.1,1; 5.6,.1; 3.8,1.3; 3.5,1.5; 3.5,1.7; 3.8,1.8;
 4.8,1.5; 5.3,1.5; 5.4,1.3; 5.9,1; 6.3,.4; 6.8,.8; 6.6,1.4; 6.4,1.8; 6.2,1.9; 6.1,2;
 6.1,2.3; 6,2.4; 6.3,2.3; 6.4,2.2; 6.6,2.1; 6.3,2.6; 7.2,.2; 8.4,1; 8.6,.9; 8.8,1.1;
 8.7,1.2; 8.5,1.3; 8.1,2.4; 7.9,2.7; 7.5,2.6; 7.4,2.8; 7.7,3.1; 7.4,3.6; 8.1,3.6; 8.4,3.7;
 7.7,4.1; 7.3,3.8; 7.8,4.2; 8.8,4.3; 8.6,4.6; 8.3,4.9; 8,5.2; 7.7,4.8; 7.5,4.7; 7.3,4.6;
 7,4.4; 6.5,4; 6.5,3.8; 6.5,3.6; 5.9,3.2; 5.6,3.1; 5,2.7; 5.4,2; 4.4,2.3; 3.7,2.4;
 2.9,2.7; 2.1,2.2; 1.7,2.2; 1.8,2; 1.9,1.8; 2.1,1.9; 1.9,2.1; .7,1.3; .2,2.6; .5,3.6;
 .4,3.7; .6,3.9; .7,3.7; 1.2,3.6; 1.2,3.5; 2.1,3.3; 2.6,3.1; 3,3; 3.3,2.9; 3.8,2.8;
 4.7,3.1; 5.2,3.6; 5.5,3.3; 5.7,3.5; 6.1,3.8; 6.3,4; 6.1,4.5; 6,4.4; 5.8,4.2; 5.6,4;
 5.4,3.7; 5,3.7; 4.6,3.4; 4.3,3.4; 4.5,3.6; 4.8,3.9; 5.3,4.4; 5.4,4.2; 5.7,4.6; 5.5,4.7;
 5.3,4.8; 5,4.6; 4.6,4.2; 4.1,4.6; 4,4.3; 4,4; .7,B; 4.1,3.6; 4.1,3.5; 3.4,3.6;
 3.5,3.8; 3.2,3.5; 3.3,3.9; 2.8,4.1; 3.3,4.2; 3.4,4.5; 3.5,4.8; 3.6,5.2; 2.8,5.2; 2.8,5;
 2.5,5.1; 2.3,4.2; 1.9,4.3; 1.4,4.4; 1.5,4.7; 1.4,5.6; 2,5.6; 3.3,6.3; 2.5,7.1; 1.4,8.3;
 .3,5.6; 1.2,5.6; 3.5,7.7; 4.4,8.2; 4.8,7.4; 5.4,7.3; 6.3,7; 6.9,6.3; 7.1,6.2; 7.3,6;
 7.4,5.8; 7.1,5.5; 6.9,5.6; 6.7,5.7; 6.5,5.8; 6,6.2; 5.8,6.4; 5.3,6.8; 4.7,7.1; 4.6,6.7;
 4.4,7; 4,6.9; 3.9,6.7; 3.9,6.6; 3.6,6.6; 3.8,6.3; 3.7,5.9; 4.2,5; 4.7,4.9; 4.3,5.7;
 4.5,6.1; 4.6,6.4; 4.2,6.6; 4.2,6.2; 5.4,5.8; 5.1,6.3; 5.7,4.9; 6,5.3; 6.2,5.6; 6.5,5.5;
 6.3,5.2; 6.4,5.1; 6.6,5.1; 6.9,5.3; 6.7,5.4; L,.4; L,1; L,2.1; L,2.6; L,3.7;
 L,4.1; L,5.5; 1.3,T; 2.8,T; 4.6,T; 8.2,T; R,7.3; R,4.9; R,4.2; R,3.6;
 R,3.1; R,2.9; R,1.7; R,1.1; R,.1; 7.3,B; 6.8,B; 6.4,B; 5.6,B; 5.4,B;
 5.2,B; 4.4,B; 4.3,B; 4.1,B; 3.9,B; 3.4,B; 2.6,B; 1.9,B; 1.7,B; 5.5,6.2];
VN=size(V,1);
E=[1,2; 1,78; 1,117; 1,186; 2,3; 2,4; 2,75; 3,5; 3,218; 3,219; 4,5; 4,6; 5,6; 5,217; 6,7;
6,216; 7,8; 7,9; 7,215; 8,9; 8,17; 9,214; 10,11; 10,12; 10,213; 11,12; 11,13; 11,212; 12,17;
13,14; 13,211; 14,15; 14,16; 15,17; 15,21; 15,23; 16,25; 16,209; 16,210; 17,18; 17,20; 18,23;
18,76; 19,20; 19,22; 19,72; 20,21; 20,70; 21,22; 21,69; 22,30; 23,29; 24,25; 24,27; 24,67;
25,26; 25,208; 26,27; 26,37; 27,28; 27,43; 28,29; 28,31; 28,35; 29,35; 30,31; 30,34; 31,32;
31,33; 32,36; 32,66; 32,68; 33,34; 33,36; 34,46; 35,45; 36,65; 37,38; 37,206;37,207; 38,39;
38,42; 39,40; 39,205; 40,41; 40,204; 41,42; 41,203; 42,43; 43,44; 44,45; 44,47; 45,202;
46,47; 46,201; 47,48; 48,49; 48,64; 48,61; 49,50; 49,200; 50,51; 50,54; 51,52; 51,53; 53,55;
53,59; 54,55; 54,199; 55,56; 55,198; 56,57; 56,58; 57,58; 57,151; 58,59; 59,60; 60,61; 60,184;
61,62; 61,183; 62,63; 62,97; 63,64; 63,96; 64,65; 64,95; 65,66; 65,94; 66,67; 66,93; 67,68;
67,69; 67,91; 69,70; 70,90; 70,71; 71,72; 71,87; 71,88; 72,73; 72,74; 72,77; 73,74; 73,76;
73,77; 73,85; 74,75; 74,78; 75,76; 78,187; 79,80; 79,85; 79,188; 79,189; 80,81; 80,83; 81,82;
81,190; 82,83; 82,142; 83,84; 84,85; 84,86; 84,134; 86,87; 86,132; 87,88; 88,89; 88,122;
89,90; 89,120; 90,91; 90,104; 91,92; 91,103; 92,93; 92,101; 92,102; 93,94; 94,95; 94,101;
95,96; 95,99; 96,98; 97,98; 97,177; 98,99; 98,109; 99,100; 100,101; 100,108; 102,103; 102,106;
102,108; 103,105; 104,105; 104,119; 105,106; 105,118; 106,107; 106,113; 107,108; 107,110;
107,112; 108,109; 109,110; 110,111; 110,177; 111,112; 111,169; 111,175; 112,113; 112,169;
113,114; 114,115; 114,127; 114,168; 115,116; 115,126; 116,118; 116,125; 118,119; 118,121;
119,120; 120,121; 121,123; 122,123; 122,124; 123,124; 123,125; 124,125; 124,132; 125,126;
126,127; 127,128; 127,130; 128,129; 128,167; 129,130; 129,167; 130,131; 131,132; 131,137;
132,133; 133,134; 133,135; 134,135; 135,136; 136,137; 136,139; 136,142; 137,138; 138,139;
138,165; 138,167; 139,140; 139,143; 140,141; 140,193; 141,142; 141,192; 143,144; 143,162;
143,194; 144,145; 144,195; 145,146; 145,159; 146,147; 146,158; 147,148; 147,157; 147,196;
148,149; 148,155; 149,150; 149,154; 149,197; 150,151; 150,153; 151,152; 152,153; 152,184;
153,154; 153,185; 154,155; 154,180; 155,156; 155,179; 156,157; 156,175; 157,220; 158,159;
158,176; 158,220; 159,160; 160,161; 160,172; 161,162; 161,173; 162,163; 163,164; 163,165;
164,166; 164,173; 165,166; 166,167; 166,174; 167,170; 168,169; 168,170; 170,171; 171,172;
171,174; 171,175; 172,173; 172,176; 175,179; 176,220; 177,178; 178,179; 178,181; 179,180;
180,181; 180,185; 181,182; 182,183; 183,184; 184,185];
EN=size(E,1);
Tmp=[4,6,3.4,.7; 7,8,3.8,.7; 11,12,4.2,.8; 13,14,5,.5; 15,23,5.4,1.2; 17,18,3.5,1.4;
18,23,4.9,1.1; 19,20,3.5,1.8; 19,22,4.9,1.4; 21,22,5.2,1.6; 28,35,6.5,1.7; 28,35,6.6,1.8;
30,31,5.9,2.1; 38,42,8.4,1.1; 44,45,7.6,2.6; 45,202,7.8,2.8; 46,47,7.4,2.9; 46,201,7.8,3;
50,51,7.8,3.8; 56,58,7.8,4.6; 57,58,7.6,5; 71,87,2.2,2.8; 71,87,2.4,3; 71,87,2.5,3;
72,73,1.9,2.3; 82,142,.9,5.4; 82,142,1.2,5.4; 90,91,4.3,2.6; 116,118,4.2,3.9; 122,124,2.7,3.6;
124,125,2.9,4.3; 128,129,3.2,5.3; 129,167,2.9,5.5; 131,137,2.4,5.2; 135,136,1.4,4.9;
136,139,1.4,6.4; 140,141,.3,7.1; 158,220,5.4,6.7; 161,162,4.3,7.1; 161,173,4.3,6.9];
lmp=3; lrl=1000; vx=[L,B;R,B;R,T;L,T];
c{1,2}=[218,72,85,189,(VN+1)]; c{2,2}=[218,72,17,213]; c{3,2}=[213,17,15,209];
c{4,2}=[209,15,29,206]; c{5,2}=[206,29,44,205]; c{6,2}=[205,44,202,(VN+2)];
```

```
c{7,2}=[202,44,48,49,54,199]; c{8,2}=[29,44,48,64,66]; c{9,2}=[15,29,66,69];
c{10,2}=[72,17,15,69,71]; c{11,2}=[85,72,71,123,133,135]; c{12,2}=[189,85,135,136,192];
c{13,2}=[71,69,66,106,105,123]; c{14,2}=[66,64,62,97,177,106]; c{15,2}=[64,48,49,54,150,97,62];
c{16,2}=[199,54,150,147,196,(VN+3)]; c{17,2}=[177,97,150,147,220,175,111];
c{18,2}=[123,105,106,177,111,175,166,167]; c{19,2}=[136,135,133,123,167,139];
c{20,2}=[192,136,139,140,193,(VN+4)]; c{21,2}=[193,140,139,143,194];
c{22,2}=[139,167,166,162,143]; c{23,2}=[143,162,166,175,220,147,145,144];
c{24,2}=[194,143,144,195]; c{25,2}=[195,144,145,147,196];
cn=size(c,1);
% road percolation
TmpN=size(Tmp,1); Tn=sparse(VN,VN); Tnp=[];
for i=1:TmpN,
  Tn(Tmp(i,1),Tmp(i,2))=0; Tnp{Tmp(i,1),Tmp(i,2)}=[];
end
for i=1:TmpN,
  Tn(Tmp(i,1),Tmp(i,2))=Tn(Tmp(i,1),Tmp(i,2))+1;
  Tnp{Tmp(i,1),Tmp(i,2)}=[Tnp{Tmp(i,1),Tmp(i,2)};Tmp(i,3:4)];
end
clf; hold on;
for i=1:EN,
  Tmp=[V(E(i,1),:)];
  if(Tn(E(i,1),E(i,2)))
    for j=1:Tn(E(i,1),E(i,2)),
      Tmp=[Tmp;Tnp{E(i,1),E(i,2)}(j,:)];
    end
  end
  Tmp=[Tmp;V(E(i,2),:)]; TmpN=size(Tmp,1);
  for j=1:(TmpN-1),
    plot([Tmp(j,1),Tmp((j+1),1)],[Tmp(j,2),Tmp((j+1),2)]);
  end
end
plot([L,R,R,L,L],[B,B,T,T,B]); tma=B-(T-B)/20;
plot([L,(L+lmp)],[tma,tma],'LineWidth',1.5);
tmb=strcat(num2str(lrl),' metres'); text((L+lmp+abs(tma/2)),tma,tmb);
axis equal; axis off; title(twn,'FontSize',15);
% vertices
NeVMat=sparse(VN,VN); LVMat=sparse(1,VN); UVMat=sparse(1,VN); bry=[]; EV=[];
for i=1:VN,
  EV{i,2}=[];
end
for i=1:EN,
  NeVMat(E(i,1),E(i,2))=1; NeVMat(E(i,2),E(i,1))=1;
  EV{E(i,1),2}=[EV{E(i,1),2},i]; EV{E(i,2),2}=[EV{E(i,2),2},i];
  if(V(E(i,1),1)<=L)
    LVMat(1,E(i,1))=1; bry=[bry,E(i,1)];
  end
  if(V(E(i,2),1)<=L)
    LVMat(1,E(i,2))=1; bry=[bry,E(i,2)];
  end
  if(V(E(i,1),1)>=R)
    UVMat(1,E(i,1))=1; bry=[bry,E(i,1)];
  end
  if(V(E(i,2),1)>=R)
    UVMat(1,E(i,2))=1; bry=[bry,E(i,2)];
  end
end
for i=1:VN,
  EV{i,1}=size(EV{i,2},2);
end
A=V; N=size(A,1); LMat=LVMat; UMat=UVMat; NeMat=NeVMat;
[pc,cord,tsries]=perc(N,LMat,UMat,NeMat);
% edges
NeEMat=sparse(EN,EN);
for i=1:VN,
  for j=1:(EV{i,1}-1),
    for k=(j+1):EV{i,1},
      NeEMat(EV{i,2}(j),EV{i,2}(k))=1; NeEMat(EV{i,2}(k),EV{i,2}(j))=1;
    end
  end
end
LEMat=sparse(1,EN); UEMat=sparse(1,EN);
for i=1:VN,
  if(LVMat(i))
    for j=1:EV{i,1},
      LEMat(EV{i,2}(j))=1;
    end
  end
  if(UVMat(i))
    for j=1:EV{i,1},
      UEMat(EV{i,2}(j))=1;
    end
  end
```

```
end
A=E; N=size(A,1); LMat=LEMat; UMat=UEMat; NeMat=NeEMat;
[pc,cord,tsries]=perc(N,LMat,UMat,NeMat);
% cells
v=[V;vx]; vn=size(v,1); lcm=sparse(1,cn); ucm=sparse(1,cn);
for i=1:cn,
  c{i,1}=size(c{i,2},2); tmp=ones(1,c{i,1}); c{i,3}=sparse(tmp,c{i,2},tmp,1,vn);
  for j=1:c{i,1},
    tma=c{i,2}(j);
    if(v(tma,1)>=R)
      ucm(i)=1;
    end
    if(v(tma,1)<=L)
      lcm(i)=1;
    end
  end
end
b=[]; ncm=sparse(cn,cn);
for i=1:(cn-1),
  for j=(i+1):cn,
    tmn=sum(c{i,3} & c{j,3});
    if(tmn>1)
      b=[b;i,j]; ncm(i,j)=1; ncm(j,i)=1;
    end
  end
end
bn=size(b,1); A=c; N=size(A,1); LMat=lcm; UMat=ucm; NeMat=ncm;
[pc,cord,tsries]=perc(N,LMat,UMat,NeMat);
% bonds
lbm=sparse(1,bn); ubm=sparse(1,bn);
for i=1:bn,
  if(lcm(b(i,1)) | lcm(b(i,2)))
    lbm(i)=1;
  end
  if(ucm(b(i,1)) | ucm(b(i,2)))
    ubm(i)=1;
  end
end
tmp=sparse(bn,cn);
for i=1:bn,
  tmp(i,b(i,1))=1; tmp(i,b(i,2))=1;
end
nbm=sparse(bn,bn);
for i=1:(bn-1),
  for j=(i+1):bn,
    tmn=sum(tmp(i,:) & tmp(j,:));
    if(tmn)
      nbm(i,j)=1; nbm(j,i)=1;
    end
  end
end
A=b; N=size(A,1); LMat=lbm; UMat=ubm; NeMat=nbm;
[pc,cord,tsries]=perc(N,LMat,UMat,NeMat);
% plot max clusters
tma=zeros(1,N); tmb=zeros(1,N);
for i=1:N,
  for j=1:2,
    tsries{2,j}=zeros(1,N); tmn=size(tsries{1,j}{i,1},1);
    tmp=[];
    for k=1:tmn,
      tmq=[tsries{1,j}{i,1}{k,1}]; tmp=[tmp,tmq];
    end
    switch j
      case 1
        tma(i)=max(tmp);
      case 2
        tmb(N-i+1)=max(tmp);
    end
  end
end
clf; tmp=1:N; tmp=tmp/N; axes('FontSize',13);
plot(tmp,tma,'LineWidth',2); hold on; plot(tmp,tmb);
axis square; axis([0,1,0,N]); xlabel('p','FontSize',15);
ylabel('Size of the largest cluster','FontSize',15);
% plot area
clf; hold on;
for i=1:cn,
  tmp=[]; tma=c{i,2};
  for j=1:c{i,1},
    tmp=[tmp;v(tma(j),:)];
  end
  tmp=[tmp;tmp(1,:)]; plot(tmp(:,1),tmp(:,2));
```

```
end
plot([L,R,R,L,L],[B,B,T,T,B]); tma=B-(T-B)/20;
plot([L,(L+lmp)],[tma,tma],'LineWidth',1.5);
tmb=strcat(num2str(lrl),' metres'); text((L+lmp+abs(tma/2)),tma,tmb);
axis equal; axis off; twm=strcat(twn,' (fire control area)');
title(twm,'FontSize',15);
```

§ A.26 Volume, surface area, cell- and face perimeters

```
% vareac.m, vareab.m transformed, (c) Kit Tiyapan, 8th December 2002
clear all; rand('state',sum(100*clock)); cnm=100; cell=rand(cnm,3);
[vtc,tmp]=voronoin(cell); vtc(1,:)=[9,9,9]; vtc=-.5+vtc; vtcn=size(vtc,1); vca=[];
for i=1:cnm,
  vca{i,2}=tmp{i}; vca{i,1}=size(tmp{i},2);
end
vfrm=ones(1,vtcn);
for i=1:vtcn,
  if(max(abs(vtc(i,:))) > 0.5)
    vfrm(i)=0;
  end
end
frm =ones(cnm, 1);
for i=1:cnm,
  tmp=1;
  for j=1:vca{i,1},
    tma=vca{i,2}(j);
    if(~vfrm(tma))
      tmp=0;
    end
  end
  if(~tmp)
    frm(i)=0;
  end
end
vc=[]; cnt=0;
for i=1:cnm,
  if(frm(i))
    cnt=cnt+1; vc{cnt,2}=vca{i,2}; vc{cnt,1}=vca{i,1}; vca{i,3}=cnt;
  else
    vca{i,3}=0;
  end
end
cn=size(vc,1);
% cell volume
tmh=[]; tmk=[]; tmu=0;
for xpx=1:-.1:.1,
  tmu=tmu+1; tmv=0; tmk=[];
  for ypx=1:-.1:.1,
    tmz=[xpx*vtc(:,1),ypx*vtc(:,2),vtc(:,3)]; tmv=tmv+1; tms=[];
    for i=1:cn,
      tmw=0; tmp=[];
      for j=1:vc{i,1},
        tmp=[tmp;tmz(vc{i,2}(j),:)];
      end
      tmd=delaunayn(tmp); tmn=size(tmd,1);
      for j=1:tmn,
        tmq=[];
        for k=1:4,
          tma=tmp(tmd(j,k),:); tmq=[tmq;tma];
        end
        tmw=tmw+(abs(det([tmq,ones(4,1)])))/6;
      end
      vc{i,3}=tmw; tms=[tms,tmw];
    end
    tmk=[tmk,ypx]; vmn(tmu,tmv)=mean(tms); vsd(tmu,tmv)=std(tms);
  end
  tmh=[tmh,xpx];
end
vmn=vmn/vmn(1,1); vsd=vsd/vsd(1,1);
[tmc,tmd]=contour(tmh,tmk,vmn,10);
[tmc,tmd]=contour(tmh,tmk,vsd,10);
% surface area
tmh=[]; tmk=[]; tmu=0;
for xpx=1:-.1:.1,
  tmu=tmu+1; tmv=0; tmk=[];
  for ypx=1:-.1:.1,
    tmz=[xpx*vtc(:,1),ypx*vtc(:,2),vtc(:,3)]; tmv=tmv+1; tmi=[];
    for i=1:cn,
      tmp=[];
      for j=1:vc{i,1},
        tmp=[tmp;tmz(vc{i,2}(j),:)];
      end
      tmq=convhulln(tmp); tmn=size(tmq,1); tmr=0;
      for j=1:tmn,
        tmx=[];
        for k=1:3,
          tmx=[tmx;tmp(tmq(j,k),:)];
        end
        tma=sum((tmx(1,:)-tmx(2,:)).^2).^0.5; tmb=sum((tmx(1,:)-tmx(3,:)).^2).^0.5;
        tmc=sum((tmx(2,:)-tmx(3,:)).^2).^0.5; tms=(tma+tmb+tmc)/2;
        tmr=tmr+sqrt(tms*(tms-tma)*(tms-tmb)*(tms-tmc));
      end
      vc{i,5}=tmr; tmi=[tmi,tmr];
    end
    tmk=[tmk,ypx]; mnm(tmu,tmv)=mean(tmi); sdm(tmu,tmv)=std(tmi);
  end
  tmh=[tmh,xpx];
end
mnm=mnm/mnm(1,1); sdm=sdm/sdm(1,1);
[tmc,tmd]=contour(tmh,tmk,mnm,10);
[tmc,tmd]=contour(tmh,tmk,sdm,10);
% perimeter
tmx=[]; tmu=0;
for xpx=1:-.1:.1,
  tmu=tmu+1; tmv=0; tmy=[];
  for ypx=1:-.1:.1,
    tmV=[xpx*vtc(:,1),ypx*vtc(:,2),vtc(:,3)]; tmv=tmv+1;
    tmj=[]; tmw=[]; tmo=ones(3,1);
    for i=1:cn,
      tmp=[];
      for j=1:vc{i,1},
        tma=tmV(vc{i,2}(j),:); tmb=vc{i,2}(j); tmp=[tmp;tma,tmb];
      end
      tmh=convhulln(tmp(:,1:3)); tmn=size(tmh,1); tmk=[];
      for j=1:tmn,
        tmq=[];
        for k=1:3,
          tma=tmp(tmh(j,k),1:3); tmq=[tmq;tma];
        end
        tma=det([tmo,tmq(:,2),tmq(:,3)]); tmb=det([tmq(:,1),tmo,tmq(:,3)]);
        tmc=det([tmq(:,1),tmq(:,2),tmo]); tmd=det(tmq);
        tmk=[tmk;tma/tmd,tmb/tmd,tmc/tmd];
      end
      tml=ones(tmn); tmg=[]; cnt=0;
      for j=1:(tmn-1),
        if(tml(j))
          cnt=cnt+1; tmg{cnt,2}=j;
          for k=(j+1):tmn,
            if(tml(k))
              tmd=abs(tmk(j,:)-tmk(k,:)); tme=1e-6;
              if((tmd(1)<tme) & (tmd(2)) & (tmd(3)))
                tmg{cnt,2}=[tmg{cnt,2},k]; tml(k)=0;
              end
            end
          end
        end
      end
      tmn=size(tmg,1);
      for j=1:tmn,
        tmg{j,1}=size(tmg{j,2});
      end
      tmf=[];
      for j=1:tmn,
        tma=sparse(vtcn,vtcn);
        tmz=[];
        for k=1:tmg{j,1},
          tmb=[];
          for l=1:3,
            tmi=tmh(tmg{j,2}(k),l); tmc=tmp(tmi,4); tmb=[tmb,tmc];
          end
          tmb=sort(tmb); tma(tmb(1),tmb(2))=tma(tmb(1),tmb(2))+1;
          tma(tmb(1),tmb(3))=tma(tmb(1),tmb(3))+1;
          tma(tmb(2),tmb(3))=tma(tmb(2),tmb(3))+1;
        end
        [tmb,tmc,tmd]=find(tma); tmm=max(size(tmb));
        for k=1:tmm,
          if(~(1-tmd(k)))
            tmz=[tmz;tmb(k),tmc(k)];
          end
        end
        tmm=size(tmz);
        tmt=0;
        for k=1:tmm,
          tmq=tmV(tmz(k,1),:); tmr=tmV(tmz(k,2),:);
          tms=sum((tmq-tmr).^2).^0.5; tmt=tmt+tms;
        end
        tmf=[tmf,tmt];
      end
      tma=sum(tmf)/2; tmb=mean(tmf); vc{i,4}=tma; vc{i,6}=tmb; tmj=[tmj,tma]; tmw=[tmw,tmb];
    end
    tmy=[tmy,ypx]; prm(tmu,tmv)=mean(tmj); prs(tmu,tmv)=std(tmj);
    pfm(tmu,tmv)=mean(tmw); pfs(tmu,tmv)=std(tmw);
  end
  tmx=[tmx,xpx];
end
prm=prm/prm(1,1); prs=prs/prs(1,1);
[tmc,tmd]=contour(tmx,tmy,prm,10);
[tmc,tmd]=contour(tmx,tmy,prs,5);
[tmc,tmd]=contour(tmx,tmy,pfm,10);
[tmc,tmd]=contour(tmx,tmy,pfs,6);
% area of face
tmx=[]; tmu=0;
for XCompression=1:-.1:.1,
  tmu=tmu+1; tmv=0; tmy=[];
  for YCompression=1:-.1:.1,
    tmV=[XCompression*Vertices(:,1),YCompression*Vertices(:,2),Vertices(:,3)];
    tmv=tmv+1; tmo=ones(3,1); afm=[]; afs=[];
    for i=1:cn,
      tmp=[];
      for j=1:vc{i,1},
        tma=tmV(vc{i,2}(j),:); tmb=vc{i,2}(j); tmp=[tmp;tma,tmb];
      end
      tmh=convhulln(tmp(:,1:3)); tmn=size(tmh,1); tmk=[];
      for j=1:tmn,
        tmq=[];
        for k=1:3,
          tma=tmp(tmh(j,k),1:3); tmq=[tmq;tma];
        end
        tma=det([tmo,tmq(:,2),tmq(:,3)]); tmb=det([tmq(:,1),tmo,tmq(:,3)]);
        tmc=det([tmq(:,1),tmq(:,2),tmo]); tmd=det(tmq); tmk=[tmk;tma/tmd,tmb/tmd,tmc/tmd];
      end
      tml=ones(tmn); tmg=[]; cnt=0;
      for j=1:(tmn-1),
        if(tml(j))
          cnt=cnt+1; tmg{cnt,2}=j;
          for k=(j+1):tmn,
            if(tml(k))
              tmd=abs(tmk(j,:)-tmk(k,:)); tme=1e-6;
              if((tmd(1)<tme) & (tmd(2)) & (tmd(3)))
                tmg{cnt,2}=[tmg{cnt,2},k]; tml(k)=0;
              end
            end
          end
        end
      end
      tmn=size(tmg,1);
      for j=1:tmn,
        tmg{j,1}=size(tmg{j,2});
      end
      tmb=[];
      for j=1:tmn,
        tma=0;
        for k=1:tmg{j,1},
          tmc=[];
          for l=1:3,
            tmi=tmh(tmg{j,2}(k),l); tmd=tmp(tmi,1:3); tmc=[tmc;tmd];
          end
          tmd=sum((tmc(1,:)-tmc(2,:)).^2).^0.5; tme=sum((tmc(1,:)-tmc(3,:)).^2).^0.5;
          tmf=sum((tmc(2,:)-tmc(3,:)).^2).^0.5; tms=(tmd+tme+tmf)/2;
          tmj=sqrt(tms*(tms-tmd)*(tms-tme)*(tms-tmf)); tma=tma+tmj;
        end
        tmb=[tmb,tma];
      end
      tma=mean(tmb); tmc=std(tmb); afm=[afm,tma]; afs=[afs,tmc];
    end
    afmm(tmu,tmv)=mean(afm); afms(tmu,tmv)=std(afm); afsm(tmu,tmv)=mean(afs);
    afss(tmu,tmv)=std(afs); tmy=[tmy,YCompression];
  end
  tmx=[tmx,XCompression];
end
afmmn=afmm/afmm(1,1); afmsn=afms/afms(1,1); afsmn=afsm/afsm(1,1); afssn=afss/afss(1,1);
[tmc,tmd]=contour(tmx,tmy,afmmn,10);
[tmc,tmd]=contour(tmx,tmy,afmsn,10);
[tmc,tmd]=contour(tmx,tmy,afsmn,10);
```

```
[tmc,tmd]=contour(tmx,tmy,afssn,10);
```

```
% varea.m      by Kittisak N. Tiyapan, 26th April, 2001
echo off; clear all; format short g; more off;
path(path,'/home/mjkvjkt/vn');
for xcp=1:-.1:.1,
  for ycp=1:-.1:.1,
    pt1 =fopen('/home/mjkvjkt/vn/vertices.dat','r');
    sc1 =fscanf(pt1, '%d', 4); dim =sc1(1,1); vnum =sc1(2,1);
    cnum =sc1(3,1); sc2 =fscanf(pt1, '%f', [dim, vnum]);
    vtc =sc2'; voc =sparse(cnum, vnum); frm =ones(cnum, 1); vfrm=ones(vnum,1);
    for i=1:cnum,
      sc1 =fscanf(pt1, '%d', 1);
      for j=1:sc1,
        sc2 =fscanf(pt1, '%d', 1); Num =sc2+1; voc(i,Num) =1;
        if ( max(abs(vtc(Num, :))) > 0.5 )
          frm(i,1) =0; vfrm(Num,1)=0;
        end
      end
    end
    vtc =([xcp*ones(vnum,1),ycp*ones(vnum,1),ones(vnum,1)]).*vtc;
    fcnm =sum(frm); vpc =full(sum(voc, 2)); [a,b,vpcn] =find(vpc.*frm);
    vnm =sparse(cnum, cnum); fnm =sparse(cnum, cnum);
    for i=1:(cnum-1),
      for j=(i+1):cnum,
        nmsh =sum(and(voc(i,:), voc(j,:)), 2);
        if (nmsh >= 3)
          fnm(i,j) =1; fnm(j,i) =1;
        end
        if (nmsh >= 1)
          vnm(i,j) =1;  vnm(j,i) =1;
        end
      end
    end
    vnpc =full(sum(vnm,2)); [a,b,vncn] =find(vnpc.*frm); fnpc =full(sum(fnm,2));
    [a,b,fncn] =find(fnpc.*frm); [c1of, c1of] =find(fnm); cofn =size(c1of,1);
    fn =cofn/2; vof =sparse(fn, vnum); ffrm =zeros(fn,1); xcof =zeros(fn,1); fcnt =0;
    for i=1:cofn,
      if (c1of(i,1) < c1of(i,1))
        fcnt =fcnt+1; vof(fcnt,:) =and( voc(c1of(i,1),:), voc(c1of(i,1),:) );
        xcof(fcnt,1) =i;
        if (frm(c1of(i,1),1)==1 | frm(c1of(i,1),1)==1)
          ffrm(fcnt,1)=1;
        end
      end
    end
    vpf=sum(vof,2); [a,b,vfin]=find(vpf.*ffrm); fcmx =max(fnpc);
    odvf =sparse(fcnt,fcmx); fdfm =find(frm); vofn=[];
    aoc =sparse(cnum,cnum); aof =sparse(fcnt,1); fppr =[];
    for i=1:fcnt,
      vthf =find(vof(i,:)); vthfn =size(vthf,2); vofn=[vofn;vthfn]; vdthf =[];
      for j=1:vthfn,
        vdthf =[vdthf; [vtc(vthf(1,j),:),vthf(1,j)]];
      end
      vspn =size(vdthf,1);
      if (vdthf(1,1)<-10 | vspn==3)
        odvc =vdthf;
      else
        X1=vdthf(1,1); X2=vdthf(2,1); X3=vdthf(3,1); Y1=vdthf(1,2);
        Y2=vdthf(2,2); Y3=vdthf(3,2); Z1=vdthf(1,3); Z2=vdthf(2,3); Z3=vdthf(3,3);
        A=det([1,Y1,Z1;1,Y2,Z2;1,Y3,Z3]); B=det([X1,1,Z1;X2,1,Z2;X3,1,Z3]);
        C=det([X1,Y1,1;X2,Y2,1;X3,Y3,1]); D= -det([X1,Y1,Z1;X2,Y2,Z2;X3,Y3,Z3]);
        FarX=10; FarY=10; FarZ= (-A*FarX-B*FarY-D)/C;
        dX1=X1-FarX; dY1=Y1-FarY; dZ1=Z1-FarZ; V1=[dX1;dY1;dZ1];
        D1=sqrt(dX1*dX1+dY1*dY1+dZ1*dZ1); dist=[D1]; gMax=0;
        JgMax=0; VJs=[dX1;dY1;dZ1];
        for j=2:vspn,
          dXj=vdthf(j,1)-FarX; dYj=vdthf(j,2)-FarY; dZj=vdthf(j,3)-FarZ;
          Vj=[dXj;dYj;dZj]; VJs=[VJs,Vj]; dist=[dist;sqrt(dXj*dXj+dYj*dYj+dZj*dZj)];
          Dj=dist(j,1); gJ=acos((dot(V1,Vj))/(D1*Dj));
          if(gJ>gMax)
            JgMax=j; gMax=gJ;
          end
        end
        g=[]; gOppPt=0; jgop=0; Vr=VJs(:,JgMax);
        for j=1:vspn,
          Vj=VJs(:,j); Dj=dist(j,1); Dr=dist(JgMax,1);
          gJ=acos((dot(Vj,Vr))/(Dr*Dj)); g=[g;gJ];
          if(gJ>gOppPt)
            jgop=j; gOppPt=gJ;
          end
        end
    ddagl=zeros(vspn,1);
    for j=1:vspn,
      if(j==JgMax | j==jgop)
        ddagl(j,1)=dist(j,1);
      else
        gRatio=g(j,1)/gOppPt; Xr=vdthf(JgMax,1); Yr=vdthf(JgMax,2);
        Zr=vdthf(JgMax,3); XOpp=vdthf(jgop,1); YOpp=vdthf(jgop,2);
        ZOpp=vdthf(jgop,3); Xdl=Xr+gRatio*(XOpp-Xr); Ydl=Yr+gRatio*(YOpp-Yr);
        Zdl=Zr+gRatio*(ZOpp-Zr); dXdl=Xdl-FarX; dYdl=Ydl-FarY; dZdl=Zdl-FarZ;
        ddagl(j,1)=sqrt(dXdl*dXdl+dYdl*dYdl+dZdl*dZdl);
      end
    end
    bck1=[]; bck2=[];
    for j=1:vspn,
      if (dist(j,1)>ddagl(j,1))
        bck1=[bck1;[vdthf(j,:),g(j,1)]];
      elseif (dist(j,1)<ddagl(j,1))
        bck2=[bck2;[vdthf(j,:),g(j,1)]];
      end
    end
    if(~isempty(bck1))
      sdbck=sortrows(bck1,5); odvc=[vdthf(JgMax,:);sdbck(:,1:4);vdthf(jgop,:)];
    else
      odvc=[vdthf(JgMax,:);vdthf(jgop,:)];
    end
    if(~isempty(bck2))
      sdbck=sortrows(bck2,5);
      for j=size(sdbck,1):-1:1,
        odvc=[odvc;sdbck(j,1:4)];
      end
    end
  end
  odvdthf=odvc; odVnl =odvdthf(:,4); odvf(i,1:vthfn) =odVnl';
  odVcd=odvdthf(:,1:3); vthfn =size(odVcd,1); stp =floor(vthfn/3);
  ndpt =1+stp; rdpt =1+2*stp;
  nvec = cross((odVcd(ndpt,:)-odVcd(1,:)),(odVcd(rdpt,:)-odVcd(1,:)));
  clstv =[odVcd; odVcd(1,:)]; Xsq =nvec(1,1)*nvec(1,1); Ysq =nvec(1,2)*nvec(1,2);
  Zsq =nvec(1,3)*nvec(1,3); nvecn =sqrt(Xsq+Ysq+Zsq); nizednV =nvec/nvecn; xps =0;
  for j=1:vthfn,
    xps =xps+cross(clstv(j,:),clstv((j+1),:));
  end
  fra =(abs(dot(nizednV, xps)))/2; nvc =ones(3,1); cdmtx =[odVcd(1,:); odVcd(ndpt,:);
  odVcd(rdpt,:)]; Aprm =det([nvc, cdmtx(:,2), cdmtx(:,3)]);
  Bprm =det([cdmtx(:,1),  nvc, cdmtx(:,3)]); Cprm =det([cdmtx(:,1), cdmtx(:,2), nvc]);
  Dprm =det([cdmtx(:,1), cdmtx(:,2), cdmtx(:,3)]); pprm=[Aprm, Bprm, Cprm, Dprm];
  athf=fra; aof(i,1) =athf; fppr =[fppr; pprm];
  aoc(c1of(xcof(i),1),c1of(xcof(i),1))=athf; aoc(c1of(xcof(i),1), c1of(xcof(i),1))=athf;
end
facn=aof.*ffrm; [fn,b,afn]=fnd(facn); mafn=mean(afn); mnafn=afn/mafn;
[a,b,sacin]=fnd(frm.*sum(aoc,2)); msacin=mean(sacin); mnsacin=sacin/msacin;
cdsAndN =zeros(cnum,4); [cVect,VerticeVect] =fnd(voc); CVPairsAmount =size(cVect);
mdc =[];
for i=1:CVPairsAmount,
  cdsAndN(cVect(i),:)=[cdsAndN(cVect(i),1:3)+vtc(cVect(i),:),cdsAndN(cVect(i),4)+1];
end
for i=1:cnum,
  mdc=[mdoc;[cdsAndN(i,1),cdsAndN(i,2),cdsAndN(i,3)]/cdsAndN(i,4)];
end
Vc =zeros(cnum,1);
for i=1:fcnt,
  if (ffrm(i,1)==1)
    A =fppr(i,1); B =fppr(i,2); C =fppr(i,3); D =fppr(i,4);
    Denom =sqrt(A*A+B*B+C*C); c1 =c1of(xcof(i),1);
    if (frm(c1,1)==1)
      Xp1 =mdc(c1,1); Yp1 =mdc(c1,2); Zp1 =mdc(c1,3);
      H1 =(A*Xp1+B*Yp1+C*Zp1+D)/Denom; V1 =abs((aof(i,1)*H1)/3); Vc(c1) =Vc(c1)+V1;
    end
    c2 =c1of(xcof(i),1);
    if (frm(c2,1)==1)
      Xp2 =mdc(c2,1); Yp2 =mdc(c2,2); Zp2 =mdc(c2,3);
      H2 =(A*Xp2+B*Yp2+C*Zp2+D)/Denom; V2 =abs((aof(i,1)*H2)/3); Vc(c2) =Vc(c2)+V2;
    end
  end
end
sVc =sum(Vc,2); Vct=sum(sVc); Vccb=Vct/fcnm; sdccb=Vccb.^(1/3);
sdaccb=sdccb*sdccb; saccb=sdaccb*6; Facepccb=sdccb*4; cpccb=sdccb*12;
[a,b,sVcin] =fnd(sVc.*frm); msVcin=mean(sVcin); mnVcin=sVcin/msVcin;
cbnVcin=sVcin/Vccb; cbnafn=afn/sdaccb; cbnsacin=sacin/saccb;
xthc=[]; Emtx=sparse(vnum,vnum); Elmtx=sparse(vnum,vnum);
pFace=sparse(fcnt,1); poc=sparse(cnum,1);
%figure(1); clf; hold on;
for i=1:fcnm,
      thc=fdfm(i,1);
      for j=1:cofn,
        if(c1of(j,1)==thc | c1of(j,1)==thc)
          xthc=[xthc;j];
        end
      end
      Sizexthc=size(xthc,1);
      Fthc=[];
      for j=1:Sizexthc,
        for k=1:fcnt,
          if(xthc(j,1)==xcof(k,1))
            Fthc=[Fthc;k];
          end
        end
      end
      fthcn=size(Fthc,1); twospthc=0;
      for j=1:fthcn,
        vlstn=vofn(Fthc(j,1),1); VList=odvf(Fthc(j,1),1:vlstn);
        X=[]; Y=[]; Z=[];
        for k=1:vlstn,
          X=[X,vtc(VList(1,k),1)]; Y=[Y,vtc(VList(1,k),2)]; Z=[Z,vtc(VList(1,k),3)];
        end
        X=[X,X(1,1)]; Y=[Y,Y(1,1)]; Z=[Z,Z(1,1)]; cvlst=[VList,VList(:,1)];
        plot3(X,Y,Z,'LineWidth',1.7); pthFace=0;
        for k=1:vlstn,
          if (Emtx(cvlst(1,k),cvlst(1,(k+1)))==0)
            Emtx(cvlst(1,k),cvlst(1,(k+1)))=1; Emtx(cvlst(1,(k+1)),cvlst(1,k))=1;
            dX=X(1,(k+1))-X(1,k); dY=Y(1,(k+1))-Y(1,k); dZ=Z(1,(k+1))-Z(1,k);
            disp=sqrt(dX*dX+dY*dY+dZ*dZ);
            Elmtx(cvlst(1,k),cvlst(1,(k+1)))=disp; Elmtx(cvlst(1,(k+1)),cvlst(1,k))=disp;
          end
          pthFace=pthFace+Elmtx(cvlst(1,k),cvlst(1,(k+1)));
        end
        pFace(Fthc(j,1),1)=pthFace; twospthc=twospthc+pthFace;
      end
      poc(thc,1)=twospthc/2;
    end
    [a,b,pfin]=find(pFace); mpfin=mean(pfin); mnpfin=pfin/mpfin; cbnpfin=pfin/Facepccb;
    [a,b,pcin]=find(poc); mpcin=mean(pcin); mnpcin=pcin/mpcin; cbnpcin=pcin/cpccb;
    xthc=[]; thc=14;
    for i=1:cofn,
      if(c1of(i,1)==thc | c1of(i,1)==thc)
        xthc=[xthc;i];
      end
    end
    Sizexthc=size(xthc,1); Fthc=[];
    for i=1:Sizexthc,
      for j=1:fcnt,
        if(xthc(i,1)==xcof(j,1))
          Fthc=[Fthc;j];
        end
      end
    end
    NumFthc=size(Fthc,1);
    for i=1:NumFthc,
      vlstn=vofn(Fthc(i,1),1); VList=odvf(Fthc(i,1),1:vlstn); X=[]; Y=[]; Z=[];
      for j=1:vlstn,
        X=[X,vtc(VList(1,j),1)]; Y=[Y,vtc(VList(1,j),2)]; Z=[Z,vtc(VList(1,j),3)];
      end
      X=[X,X(1,1)]; Y=[Y,Y(1,1)]; Z=[Z,Z(1,1)]; fill3(X,Y,Z,i);
    end
    box on; axis equal; view(-20,10); title('A Voronoi c with six others','FontSize',12);
    xlabel('x','FontSize',11); ylabel('y','FontSize',11); zlabel('z','FontSize',11);
    Numfin=sum(ffrm,1);
  end;
end;
```

§ A.27 Volume in higher dimensions

```
% volnd.m, higher-d volumes, (c) Kit Tiyapan, 9th December 2002
clear all; rand('state',sum(100*clock)); cna=500;
dim=5; c=rand(cna,dim); [v,tmp]=voronoin(c); vn=size(v,1);
for i=1:cna,
  vca{i,2}=tmp{i}; vca{i,1}=size(tmp{i},2);
end
vin=sparse(1,vn);
for i=1:vn,
  if((max(v(i,:))<1) & (min(v(i,:))>0))
    vin(i)=1;
  end
end
cin=sparse(1,cna);
for i=1:cna,
  tmn=1;
  for j=1:vca{i,1},
    tma=vin(vca{i,2}(j));
    if(~tma)
      tmn=0;
    end
  end
  if(tmn)
    cin(i)=1;
  end
end
tmv=[]; tmz=[];
tmm=factorial(dim);
for i=1:cna,
  if(cin(i))
    tmp=[];
    for j=1:vca{i,1},
      tma=v(vca{i,2}(j),:); tmp=[tmp;tma];
    end
    tmd=delaunayn(tmp); tmn=size(tmd,1); tms=0;
    for j=1:tmn,
      tmq=[];
      for k=1:(dim+1),
        tma=tmp(tmd(j,k),:); tmq=[tmq;tma];
      end
      tmq=[tmq,ones((dim+1),1)]; tma=abs(det(tmq))/tmm; tms=tms+tma;
    end
    tmv=[tmv,tms]; tma=max(tmp,[],1); tmb=min(tmp,[],1);
    tmc=prod(tma-tmb); tmz=[tmz,tmc];
  end
end
tma=tmv./tmz; tmm=sum(vin); tmn=sum(cin); vrm=mean(tma); vrs=std(tma);
% test volume formula
clear all; dim=7; d=3; tmp=[0,d]';
for i=2:dim,
  tmp=[[zeros(2^(i-1),1);d*ones(2^(i-1),1)],[tmp;tmp]];
end
v=tmp; tmd=delaunayn(v); tmn=size(tmd,1); tms=0; tmm=1;
for i=2:dim,
  tmm=tmm*i;
end
for i=1:tmn,
  tmp=[];
  for j=1:(dim+1),
    tma=v(tmd(i,j),:); tmp=[tmp;tma];
  end
  tmc=[tmp,ones((dim+1),1)]; tma=abs(det(tmc))/tmm; tms=tms+tma;
end
```

§ A.28 Regular lattices in three dimensions

```
% trr.m, regular 3-d tesselation, (c) Kit Tiyapan, 16 December 2002
qn=size(q,1); in2n=size(in2,1); in3n=size(in3,1); in5n=size(in5,1);
p=[]; p{1,1}=[o;o]'; p{1,2}=sz; r=[]; s=[]; t=[];
for i=1:sz,
  r(1,i)=dx(m(i)); s(1,i)=dy(n(i)); t(1,i)=dz(z(i));
end
p{1,3}=r'; p{1,4}=s'; p{1,5}=t'; iin=size(ii,1); tmp=ones(iin,1);
tma=sparse(tmp,ii(:,1),tmp,1,sz); tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{2,1}=tmp; p{2,2}=cnt; p{2,3}=tmb; p{2,4}=tmc; p{2,5}=tmd; iiin=size(iii,1);
tmp=ones(iiin,1); tma=sparse(tmp,iii(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{3,1}=tmp; p{3,2}=cnt; p{3,3}=tmb; p{3,4}=tmc; p{3,5}=tmd; iv=[ii,2*ones(iin,1)];
for i=1:iiin,
  tmp=0;
  for j=1:iin,
    if(~(iii(i,1)-ii(j,1)))
      tmp=1;
    end
  end
  if(~tmp)
    iv=[iv;iii(i,:),3];
  end
end
ivn=size(iv,1); tmp=ones(ivn,1); tma=sparse(tmp,iv(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{4,1}=tmp; p{4,2}=cnt; p{4,3}=tmb; p{4,4}=tmc;
p{4,5}=tmd; vn=size(v,1); tmp=ones(vn,1);
tma=sparse(tmp,v(:,1),tmp,1,sz); tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{5,1}=tmp; p{5,2}=cnt; p{5,3}=tmb; p{5,4}=tmc; p{5,5}=tmd; vi=[ii,2*ones(iin,1)];
for i=1:vn,
  tmp=0;
  for j=1:iin,
    if(~(v(i,1)-ii(j,1)))
      tmp=1;
    end
  end
  if(~tmp)
    vi=[vi;v(i,:),5];
  end
end
vin=size(vi,1); tmp=ones(vin,1); tma=sparse(tmp,vi(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{6,1}=tmp; p{6,2}=cnt; p{6,3}=tmb; p{6,4}=tmc; p{6,5}=tmd; vii=[iii,3*ones(iiin,1)];
for i=1:vn,
    tmp=0;
  for j=1:iiin,
    if(~(v(i,1)-iii(j,1)))
      tmp=1;
    end
  end
  if(~tmp)
    vii=[vii;v(i,:),5];
  end
end
```

```
viin=size(vii,1); tmp=ones(viin,1); tma=sparse(tmp,vii(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{7,1}=tmp; p{7,2}=cnt; p{7,3}=tmb; p{7,4}=tmc; p{7,5}=tmd; viii=iv;
for i=1:vn,
  tmp=0;
  for j=1:ivn,
    if(~(v(i,1)-iv(j,1)))
      tmp=1;
    end
  end
  if(~tmp)
    viii=[viii;v(i,:),5];
  end
end
viiin=size(viii,1); tmp=ones(viiin,1); tma=sparse(tmp,viii(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{8,1}=tmp; p{8,2}=cnt; p{8,3}=tmb; p{8,4}=tmc; p{8,5}=tmd; map=[]; tmp=ones(sz,1);
map{1,1,1}=sparse(p{1,1}(:,1),tmp,p{1,1}(:,2),sz,1); cnt=p{1,2}; e=q;
V=[p{1,3},p{1,4},p{1,5}]; tmp=ones(p{2,2},1);
for i=2:nx,
  map{i,1,1}=sparse(p{2,1}(:,1),tmp,cnt*tmp+p{2,1}(:,2),sz,1);
  for k=1:iin,
    map{i,1,1}(ii(k,1),1)=map{(i-1),1,1}(ii(k,2),1);
  end
  cnt=cnt+p{2,2}; % +iin;
  for k=1:qn,
    e=[e;map{i,1,1}(q(k,1)),map{i,1,1}(q(k,2))];
  end
  for k=1:in2n,
    e=[e;map{i,1,1}(in2(k,1)),map{(i-1),1,1}(in2(k,2))];
  end
  V=[V;(i-1)*dim1*tmp+p{2,3},p{2,4},p{2,5}];
end
tmp=ones(p{3,2},1);
for j=2:ny,
  map{1,j,1}=sparse(p{3,1}(:,1),tmp,cnt*tmp+p{3,1}(:,2),sz,1);
  for k=1:iiin,
    map{1,j,1}(iii(k,1),1)=map{1,(j-1),1}(iii(k,2),1);
  end
  cnt=cnt+p{3,2}; % +iiin;
  for k=1:qn,
    e=[e;map{1,j,1}(q(k,1)),map{1,j,1}(q(k,2))];
  end
  for k=1:in3n,
    e=[e;map{1,j,1}(in3(k,1)),map{1,(j-1),1}(in3(k,2))];
  end
  V=[V;p{3,3},(j-1)*dim2*tmp+p{3,4},p{3,5}];
end
tmp=ones(p{4,2},1);
for i=2:nx,
  for j=2:ny,
    map{i,j,1}=sparse(p{4,1}(:,1),tmp,cnt*tmp+p{4,1}(:,2),sz,1);
    for k=1:ivn,
      if(iv(k,3)==2)
        map{i,j,1}(iv(k,1),1)=map{(i-1),j,1}(iv(k,2),1);
      else
        map{i,j,1}(iv(k,1),1)=map{i,(j-1),1}(iv(k,2),1);
      end
    end
    cnt=cnt+p{4,2}; % +ivn;
    for k=1:qn,
      e=[e;map{i,j,1}(q(k,1)),map{i,j,1}(q(k,2))];
    end
    for k=1:in2n,
      e=[e;map{i,j,1}(in2(k,1)),map{(i-1),j,1}(in2(k,2))];
    end
    for k=1:in3n,
      e=[e;map{i,j,1}(in3(k,1)),map{i,(j-1),1}(in3(k,2))];
    end
    V=[V;(i-1)*dim1*tmp+p{4,3},(j-1)*dim2*tmp+p{4,4},p{4,5}];
  end
end
```

```
tmp=ones(p{5,2},1);
for i=2:nz,
  map{1,1,i}=sparse(p{5,1}(:,1),tmp,cnt*tmp+p{5,1}(:,2),sz,1);
  for k=1:vn,
    map{1,1,i}(v(k,1),1)=map{1,1,(i-1)}(v(k,2),1);
  end
  cnt=cnt+p{5,2};
  for k=1:qn,
    e=[e;map{1,1,i}(q(k,1)),map{1,1,i}(q(k,2))];
  end
  for k=1:in5n,
    e=[e;map{1,1,i}(in5(k,1)),map{1,1,(i-1)}(in5(k,2))];
  end
  V=[V;p{5,3},p{5,4},(i-1)*dim3*tmp+p{5,5}];
end
tmp=ones(p{6,2},1);
for i=2:nx,
  for j=2:nz,
    map{i,1,j}=sparse(p{6,1}(:,1),tmp,cnt*tmp+p{6,1}(:,2),sz,1);
    for k=1:vin,
      if(~(vi(k,3)-2))
        map{i,1,j}(vi(k,1),1)=map{(i-1),1,j}(vi(k,2),1);
      else
        map{i,1,j}(vi(k,1),1)=map{i,1,(j-1)}(vi(k,2),1);
      end
    end
    cnt=cnt+p{6,2};
    for k=1:qn,
      e=[e;map{i,1,j}(q(k,1)),map{i,1,j}(q(k,2))];
    end
    for k=1:in2n,
      e=[e;map{i,1,j}(in2(k,1)),map{(i-1),1,j}(in2(k,2))];
    end
    for k=1:in5n,
      e=[e;map{i,1,j}(in5(k,1)),map{i,1,(j-1)}(in5(k,2))];
    end
    V=[V;(i-1)*dim1*tmp+p{6,3},p{6,4},(j-1)*dim3*tmp+p{6,5}];
  end
end
tmp=ones(p{7,2},1);
for i=2:ny,
  for j=2:nz,
    map{1,i,j}=sparse(p{7,1}(:,1),tmp,cnt*tmp+p{7,1}(:,2),sz,1);
    for k=1:viin,
      if(~(vii(k,3)-3))
        map{1,i,j}(vii(k,1),1)=map{1,(i-1),j}(vii(k,2),1);
      else
        map{1,i,j}(vii(k,1),1)=map{1,i,(j-1)}(vii(k,2),1);
      end
    end
    cnt=cnt+p{7,2};
    for k=1:qn,
      e=[e;map{1,i,j}(q(k,1)),map{1,i,j}(q(k,2))];
    end
    for k=1:in3n,
      e=[e;map{1,i,j}(in3(k,1)),map{1,(i-1),j}(in3(k,2))];
    end
    for k=1:in5n,
      e=[e;map{1,i,j}(in5(k,1)),map{1,i,(j-1)}(in5(k,2))];
    end
    V=[V;p{7,3},(i-1)*dim2*tmp+p{7,4},(j-1)*dim3*tmp+p{7,5}];
  end
end
tmp=ones(p{8,2},1);
for i=2:nx,
  for j=2:ny,
    for k=2:nz,
      map{i,j,k}=sparse(p{8,1}(:,1),tmp,cnt*tmp+p{8,1}(:,2),sz,1);
      for m=1:viiin,
        if(~(viii(m,3)-2))
          map{i,j,k}(viii(m,1),1)=map{(i-1),j,k}(viii(m,2),1);
        elseif(~(viii(m,3)-3))
          map{i,j,k}(viii(m,1),1)=map{i,(j-1),k}(viii(m,2),1);
        else
          map{i,j,k}(viii(m,1),1)=map{i,j,(k-1)}(viii(m,2),1);
        end
      end
      cnt=cnt+p{8,2};
      for m=1:qn,
        e=[e;map{i,j,k}(q(m,1)),map{i,j,k}(q(m,2))];
      end
      for m=1:in2n,
        e=[e;map{i,j,k}(in2(m,1)),map{(i-1),j,k}(in2(m,2))];
      end
      for m=1:in3n,
        e=[e;map{i,j,k}(in3(m,1)),map{i,(j-1),k}(in3(m,2))];
      end
      for m=1:in5n,
        e=[e;map{i,j,k}(in5(m,1)),map{i,j,(k-1)}(in5(m,2))];
      end
      V=[V;(i-1)*dim1*tmp+p{8,3},(j-1)*dim2*tmp+p{8,4},(k-1)*dim3*tmp+p{8,5}];
    end
  end
end
en=size(e,1); Vn=size(V,1);
figure(1); clf; hold on;
for i=1:en,
  plot3([V(e(i,1),1),V(e(i,2),1)],[V(e(i,1),2),V(e(i,2),2)],[V(e(i,1),3),V(e(i,2),3)]);
end
axis off; axis equal;
clf; hold on; tms=sum(nemat,2); tmp=[]; tma=17.2; % 5x5x5 units
for i=1:en,
  if(tms(i)<tma)
    tmp=[tmp;e(i,:)];
  end
end
tmn=size(tmp,1);
for i=1:tmn,
  plot3([V(tmp(i,1),1),V(tmp(i,2),1)],...
    [V(tmp(i,1),2),V(tmp(i,2),2)],[V(tmp(i,1),3),V(tmp(i,2),3)]);
end
axis off; axis equal;
% for vertices
nvmat=sparse(Vn,Vn);
for i=1:en,
  nvmat(e(i,1),e(i,2))=1; nvmat(e(i,2),e(i,1))=1;
end
A=V; N=Vn; lmat=sparse(1,N); umat=sparse(1,N); LB=min(V(:,1)); UB=max(V(:,1));
rng=UB-LB; LBv=.05*rng+LB; UBv=UB-LBv;
for i=1:Vn,
  if(V(i,1)<LBv)
    lmat(1,i)=1;
  end
  if(V(i,1)>UBv)
    umat(1,i)=1;
  end
end
nmat=nvmat; Blocked=randperm(Vn); [pc,cord,tsries]=perc(N,lmat,umat,nmat);
% for edges
evm=sparse(en,Vn);
for i=1:en,
  evm(i,e(i,1))=1; evm(i,e(i,2))=1;
end
nemat=sparse(en,en);
for i=1:Vn,
  tmp=find(evm(:,i)); tmn=size(tmp,1);
  for j=1:(tmn-1),
    for k=(j+1):tmn,
      nemat(tmp(j),tmp(k))=1; nemat(tmp(k),tmp(j))=1;
    end
  end
end
A=e; N=en; lmat=sparse(1,N); umat=sparse(1,N);
for i=1:N,
  if((V(A(i,1),1)<=LBv) | (V(A(i,2),1)<=LBv))
    lmat(1,i)=1;
  elseif((V(A(i,1),1)>=UBv) | (V(A(i,2),1)>=UBv))
    umat(1,i)=1;
  end
end
nmat=nemat; Blocked=randperm(N); [pc,cord,tsries]=perc(N,lmat,umat,nmat);
% ccp
clear all; sz=9; nx=5; ny=5; nz=5; r=1; tmp=sqrt(2)*r;
dx=[0,2*r,4*r]; dy=[0,2*r,4*r]; dz=[0,tmp,2*tmp];
dim1=max(dx); dim2=max(dy); dim3=max(dz);
q=[1,2;1,3;1,5;1,6;2,5;3,5;4,5;5,6;5,7;5,8;5,9]; m=[1,3,1,3,2,1,3,1,3];
n=[1,1,3,3,2,1,1,3,3]; z=[1,1,1,1,2,3,3,3,3];o=[1,2,3,4,5,6,7,8,9];
ii=[1,2;3,4;6,7;8,9]; iii=[1,3;2,4;6,8;7,9]; v=[1,6;2,7;3,8;4,9];
in2=[]; in3=[]; in5=[];
```

§ A.29 Effects of channelling

```
% chl.m, effect of channelling, (c) Kit Tiyapan 17 December 2002
clear all; rand('state',sum(100*clock)); can=1000; ca=rand(can,3);
[va,vca]=voronoin(ca); van=size(va,1); tmv=sparse(1,van);
for i=1:van,
  if((max(va(i,:))<1)&(min(va(i,:))>0))
    tmv(i)=1;
  end
end
vcan=[]; ca=[];
for i=1:can,
  tmn=size(vca{i},2); in=1;
  for j=1:tmn,
    tma=tmv(vca{i}(j));
    if(~tma)
      in=0;
    end
  end
  if(in)
    tmp=[];
    for j=1:tmn,
      tma=va(vca{i}(j),:); tmp=[tmp;tma];
    end
    tmd=delaunayn(tmp); tmn=size(tmd,1); tmk=[];
    for j=1:tmn,
      tma=[];
      for k=1:4,
        tmb=tmp(tmd(j,k),:); tma=[tma;tmb];
      end
      tma=sum(tma,1)/4; tmk=[tmk;tma];
    end
    tmk=sum(tmk,1)/tmn; ca=[ca;tmk];
  end
end
tma=min(min(ca)); tmb=max(max(ca)); tmr=tmb-tma; ca=(ca-tma*ones(size(ca)))/tmr;
can=size(ca,1); [va,vca]=voronoin(ca); van=size(va,1); vcan=[];
for i=1:can,
  vcan=[vcan,size(vca{i},2)];
end
cvm=[];
for i=1:can,
  tma=ones(1,vcan(i)); cvm=[cvm;sparse(tma,vca{i},tma,1,van)];
end
ncc=sparse(can,can);
for i=1:(can-1),
  for j=(i+1):can,
    tma=sum(cvm(i,:)&cvm(j,:));
    if(tma)
      ncc(i,j)=1; ncc(j,i)=1;
    end
  end
end
tmv=sparse(1,van);
for i=1:van,
  if((max(va(i,:))<1)&(min(va(i,:))>0))
    tmv(i)=1;
  end
end
tmc=sparse(1,can);
for i=1:can,
  in=1;
  for j=1:vcan(i),
    tma=tmv(vca{i}(j));
    if(~tma)
      in=0; break;
    end
  end
  if(~(in-1))
    tmc(i)=1;
  end
end
c=[]; cnt=0;
for i=1:can,
  if(tmc(i))
    cnt=cnt+1; c=[c;ca(i,:)]; tmc(i)=cnt;
  end
end
cn=size(c,1); necc=sparse(cn,cn); [tma,tmb]=find(triu(ncc)); tmn=size(tma,1);
for i=1:tmn,
  tmp=tmc(tma(i)); tmq=tmc(tmb(i));
  if(tmp&tmq)
    necc(tmp,tmq)=1; necc(tmq,tmp)=1;
  end
end
% cells
tma=min(c(:,1)); tmb=max(c(:,1)); tmr=tmb-tma; tmd=tmr*.1; lb=tma+tmd;
ub=tmb-tmd; A=c; N=size(A,1); lmat=sparse(1,N); umat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=lb)
    lmat(1,i)=1;
  elseif(A(i,1)>=ub)
    umat(1,i)=1;
  end
end
NeMat=necc; Blocked=randperm(cn); [pc,cord,tsries]=perc(N,lmat,umat,NeMat);
% cells rivulets, steepest input
b=[]; cnt=0; [tma,tmb]=find(necc); tmn=size(tma,1);
for i=1:tmn,
  cnt=cnt+1; tmp=c(tma(i),3); tmq=c(tmb(i),3);
  if(tmp>tmq)
    b(cnt,1)=tma(i); b(cnt,2)=tmb(i);
  else
    b(cnt,1)=tmb(i);b(cnt,2)=tma(i);
  end
end
bn=size(b,1);
for i=1:bn,
  tma=c(b(i,1),1:2); tmb=c(b(i,2),1:2); tmd=sum((tma-tmb).^2).^0.5;
  tma=c(b(i,1),3)-c(b(i,2),3); tmb=tma/tmd; tma=atan(tmb); b(i,3)=tma;
end
tmz=b(:,2:3); tmw=zeros(1,cn); tmx=zeros(1,cn);
for i=1:bn,
  tma=tmz(i,1);
  if(tmz(i,2)>tmw(tma))
    tmw(tma)=tmz(i,2); tmx(tma)=tmz(i,1);
  end
end
tma=sortrows([tmw',tmx'],1); tmx=tma(:,2)';
for i=1:cn,
  if(tmx(i))
    tmn=i-1; tma=tmx(i);
    break;
  end
end
tmx(1,1:tmn)=tma*ones(1,tmn); tma=min(c(:,1)); tmb=max(c(:,1)); tmr=tmb-tma; tmd=tmr*.1;
lb=tma+tmd; ub=tmb-tmd; A=c; N=size(A,1); lmat=sparse(1,N); umat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=lb)
    lmat(1,i)=1;
  elseif(A(i,1)>=ub)
    umat(1,i)=1;
  end
end
NeMat=necc; [pc,cord,tsries]=perd(N,lmat,umat,NeMat,tmx);
% cells rivulets, max. sum sign
tmw=[b(:,1),b(:,3)]; tmx=zeros(1,cn);
for i=1:bn,
  tma=tmz(i,1); tmb=tmw(i,1); tmx(tma)=tmx(tma)+tmz(i,2);
  tmx(tmb)=tmx(tmb)-tmw(i,2);
end
tma=[1:cn;tmx]'; tmx=sortrows(tma,2); tmx=tmx(:,1)'; tma=min(c(:,1));
tmb=max(c(:,1)); tmr=tmb-tma; tmd=tmr*.1; lb=tma+tmd; ub=tmb-tmd; A=c;
N=size(A,1); lmat=sparse(1,N); umat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=lb)
    lmat(1,i)=1;
  elseif(A(i,1)>=ub)
    umat(1,i)=1;
  end
end
NeMat=necc; [pc,cord,tsries]=perd(N,lmat,umat,NeMat,tmx);
% cells, combined
for i=1:bn,
  tma=c(b(i,1),:); tmb=c(b(i,2),:); tmd=sum((tma-tmb).^2).^0.5; tmd=tmd/2; b(i,4)=tmd;
end
cb=[];
for i=1:cn,
  cb{i,3}=[]; cb{i,4}=[];
end
for i=1:can,
  if(inc(i))
    tmp=[];
    for j=1:vcan(i),
      tma=va(vca{i}(j),:); tmp=[tmp;tma];
    end
    tmd=delaunayn(tmp); tmn=size(tmd,1); tmb=0;
    for j=1:tmn,
      tma=[tmp(tmd(j,1),:);tmp(tmd(j,2),:);tmp(tmd(j,3),:);tmp(tmd(j,4),:)];
      tmq=abs(det([tma,ones(4,1)]))/6; tmb=tmb+tmq;
    end
    cb{inc(i),1}=tmb;
  end
end
for i=1:bn,
  tma=b(i,1); cb{tma,3}=[cb{tma,3},i];
end
for i=1:cn,
  cb{i,2}=size(cb{i,3},2);
end
for i=1:cn,
  tmt=0;
  for j=1:cb{i,2},
    tma=b(cb{i,3}(j),3); tmt=tmt+tma;
  end
  for j=1:cb{i,2},
    tma=b(cb{i,3}(j),3)/tmt; cb{i,4}=[cb{i,4},tma];
  end
end
res=1000; map=zeros(res,res); cr=c*res; vr=va*res; z=max(max(cr))*.9; cin=zeros(1,cn);
for i=1:can,
  if(inc(i))
    cin(inc(i))=i;
  end
end
mac=sparse(res,res);
for i=1:cn,
  tmp=[];
  for j=1:vcan(cin(i)),
    tma=vr(vca{cin(i)}(j),:); tmp=[tmp;tma];
  end
  tmh=convhulln(tmp); tmn=size(tmh,1); tmj=[]; tmj{i,2}=[]; tmj{i,1}=0;
  for j=1:tmn,
    tmq=[];
    for k=1:3,
      tma=tmp(tmh(j,k),:); tmq=[tmq;tma];
    end
    tma=max(tmq(:,3)); tmb=min(tmq(:,3));
    in=0;
    if((tma>z)&(tmb<z))
      in=1; tma=[tmq(1,:);tmq(2,:)]; tmb=max(tma(:,3)); tmc=min(tma(:,3)); tmt=[];
      if((tmb>z)&(tmc<z))
        tmt{1}=tma; tma=[tmq(1,:);tmq(3,:)]; tmb=max(tma(:,3)); tmc=min(tma(:,3));
        if((tmb>z)&(tmc<z))
          tmt{2}=tma;
        else
          tmt{2}=[tmq(2,:);tmq(3,:)];
        end
      else
        tmt{1}=[tmq(1,:);tmq(3,:)]; tmt{2}=[tmq(2,:);tmq(3,:)];
      end
    end
    if(in)
      tmi=[];
      for k=1:2,
        x1=tmt{k}(1,1); x2=tmt{k}(2,1); y1=tmt{k}(1,2); y2=tmt{k}(2,2); z1=tmt{k}(1,3);
        z2=tmt{k}(2,3); x12=x2-x1; y12=y2-y1; z12=z2-z1; t=(z-z1)/z12; x=x1+x12*t;
        y=y1+y12*t; tmi=[tmi;x,y]; tmj{i,2}=[tmj{i,2};x,y]; tmj{i,1}=tmj{i,1}+1;
      end
      x1=tmi(1,1); y1=tmi(1,2); x2=tmi(2,1); y2=tmi(2,2); x12=x2-x1; y12=y2-y1;
      tmi=0.1; tma=sign(y2-y1); tms=tmi*tma; tme=abs(y2)-tmi*tma;
      for k=y1:tms:tme,
        x=round(x1+(k-y1)*x12/y12); map(x,round(k))=1;
      end
      tma=sign(x2-x1); tms=tmi*tma; tme=abs(x2)-tmi*tma;
      for k=x1:tms:tme,
        y=round(y1+(k-x1)*y12/x12); map(round(k),y)=1;
      end
      map(round(x2),round(y2))=1;
    end
  end
  if(tmj{i,1})
    tma=sum(tmj{i,2},1)/tmj{i,1};
    tma=round(tma);
    mac(tma(1),tma(2))=i;
  end
end
maq=map;
for i=1:res,
  maq(i,res)=-1; maq(i,1)=-1; maq(res,i)=-1; maq(1,i)=-1;
end
cnt=1;
cnc=[];
for j=2:(res-1),
  i=1;
  while(i<res)
    tmp=[];
    i=i+1;
    while(~maq(i,j))
      tmp=[tmp,i]; i=i+1;
    end
    tmn=size(tmp,2);
    if(tmn)
      tmc=0;
      for k=1:tmn,
        tma=maq(tmp(k),(j-1));
        if(tma-1)
          tmc=tma;   break;
        end
      end
      if(~tmc)
        cnt=cnt+1; tmc=cnt;
      end
      for k=1:tmn,
        maq(tmp(k),j)=tmc;
      end
    end
  end
end
[tmi,tmj,tmk]=find(mac);
tmn=size(tmi,1);
tml=[];
for i=1:tmn,
  tma=maq(tmi(i),tmj(i)); tml=[tml;tma];
end
sc=zeros(1,cnt);
for i=1:tmn,
  sc(tml(i))=tmk(i);
end
for i=1:res,
  for j=1:res,
    if(maq(i,j)>1)
      tma=tmk(sc(maq(i,j))); maq(i,j)=tma;
    end
  end
end
clf; hold on;
for i=1:20:res,
  tma=[(i-5),(i+5),(i+5),(i-5)];
  for j=1:10:res,
    tmb=[(j-5),(j-5),(j+5),(j+5)]; tmc=maq(i,j); fill(tma,tmb,tmc);
  end
end
axis equal;
axis off;
% map section to cell
[tma,tmb]=find(triu(necc));
tmn=size(tma,1);
tmx=[];  % top cells
for i=1:tmn,
  tmp=[cr(tma(i),:);cr(tmb(i),:)]; tmq=max(tmp(:,3)); tmr=min(tmp(:,3));
  if((tmq>z)&(tmr<z))
    tmp=sortrows(tmp,3); tmx=[tmx;tmp(1,:)];
  end
end
tmm=size(tmx,1);
tmp=[];
for i=1:cnt,
  tmp{i,2}=[]; tmp{i,1}=0;
end
tmn=100;
for i=1:res,
  for j=1:res,
    tma=maq(i,j);
    if(tma+1)
      if(tma&(tma-1)&(tmp{tma,1}<tmn))
        tmp{tma,2}=[tmp{tma,2};i,j]; tmp{tma,1}=tmp{tma,1}+1;
      end
    end
  end
end
tmy=zeros(cnt,3);  % grid from sect
for i=1:cnt,
  tma=sum(tmp{i,2},1)/tmp{i,1}; tmy(tma,:)=[tma(1),tma(2),z];
end
tmn=size(tmy,1); tmp=zeros(tmm,tmn);
for i=1:tmm,
  for j=1:tmn,
    tmp(i,j)=sum((tmx(i,:)-tmy(j,:)).^2).^0.5;
  end
end
gc=zeros(1,tmn);
for j=2:tmn,
  tma=[];
  for i=1:tmm,
    tma=[tma;i,tmp(i,j)];
  end
  tma=sortrows(tma,2); gc(j)=tma(1,1);
end
tmy=zeros(cnt,3);  % grid from sect
for i=1:20:res,
  for j=1:20:res,
    tma=maq(i,j);
    if(tma+1)
      if(~tmy(tma))
        tmy(tma,:)=[i,j,z];
      end
    end
  end
end
tmn=size(tmy,1); tmp=zeros(tmm,tmn);
for i=1:tmm,
  for j=1:tmn,
    tmp(i,j)=sum((tmx(i,:)-tmy(j,:)).^2).^0.5;
  end
end
for j=2:tmn,
  tma=[];
  for i=1:tmm,
    tma=[tma;i,tmp(i,j)];
  end
  tma=sortrows(tma,2); gc(j)=tma(1,1);
end
% look at each sectional cell
tmp=sparse(res,res); tma=8;
for i=1:res,
  for j=1:res,
    if(~(tma-map1(i,j)))
      tmp(i,j)=1;
    end
  end
end
% which link which section?
map1=map;
[tma,tmb]=find(triu(necc));
tmn=size(tma,1);
for i=1:tmn,
  tmp=[cr(tma(i),:);cr(tmb(i),:)]; tmq=max(tmp(:,3)); tmr=min(tmp(:,3));
  if((tmq>z)&(tmr<z))
    x1=tmp(1,1); x2=tmp(2,1); y1=tmp(1,2); y2=tmp(2,2); z1=tmp(1,3); z2=tmp(2,3);
    x12=x2-x1; y12=y2-y1; z12=z2-z1; t=(z-z1)/z12; x=x1+x12*t; y=y1+y12*t;
    map1(round(x),round(y))=1;
  end
end
% or this?
map1=map;
[tma,tmb]=find(triu(necc)); tmn=size(tma,1);
for i=1:tmn,
  tmp=[cr(tma(i),:);cr(tmb(i),:)];
  x1=round(tmp(1,1)); x2=round(tmp(2,1)); y1=round(tmp(1,2)); y2=round(tmp(2,2));
  map1(x1,y1)=1; map1(x2,y2)=1;
end
% test, successful, ie. all its faces completely cover the hull leaving no gaps
tst=[];
for i=1:cn,
  tmp=[];
  for j=1:vcan(cin(i)),
    tma=vr(vca{cin(i)}(j),:); tmp=[tmp;tma,vca{cin(i)}(j)];
  end
  tmh=convhulln(tmp(:,1:3)); tmn=size(tmh,1); tss=sparse(van,van);
  for j=1:tmn,
    tma=[tmp(tmh(j,1),4),tmp(tmh(j,2),4)]; tma=sort(tma);
    tss(tma(1),tma(2))=tss(tma(1),tma(2))+1; tma=[tmp(tmh(j,1),4),tmp(tmh(j,3),4)];
    tma=sort(tma); tss(tma(1),tma(2))=tss(tma(1),tma(2))+1; tma=[tmp(tmh(j,2),4),tmp(tmh(j,3),4)];
    tma=sort(tma); tss(tma(1),tma(2))=tss(tma(1),tma(2))+1;
  end
  [tma,tmb,tmc]=find(tss); tma=min(tmc);tst=[tst,tma];
end
```

§ A.30 Stereographic projection

```
% stp.m, stereographic projection, (c) Kit Tiyapan, January 2002
clear all; R=2; V=[1,-1,-1;1,-1,1;1,1,1;1,1,-1;-1,-1,-1;-1,-1,1;-1,1,1;-1,1,-1];
VN=size(V,1); E=[1,2;2,3;3,4;1,4;5,6;6,7;7,8;5,8;1,5;2,6;3,7;4,8];
EN=size(E,1); Res=100; El=[];
for i=1:EN,
  El{i,1}=[];
end
for i=1:EN,
  x1=V(E(i,1),1); y1=V(E(i,1),2); z1=V(E(i,1),3); x2=V(E(i,2),1); y2=V(E(i,2),2);
  z2=V(E(i,2),3); dx=x2-x1; dy=y2-y1; dz=z2-z1; d=sqrt(dx*dx+dy*dy+dz*dz);
  for t=0:(1/Res):1,
    x=x1+dx*t; y=y1+dy*t; z=z1+dz*t; El{i,1}=[El{i,1};x,y,z];
  end
end
V1=[]; V2=[]; H=[];
for i=1:VN,
  x=V(i,1); y=V(i,2);  z=V(i,3); d=sqrt(x*x+y*y+z*z); t=R/d;
  x1=x*t; y1=y*t; z1=z*t; V1=[V1;x1,y1,z1]; x2=0; y2=0;
  if(z1>=0)
    z2=-R; H=[H;0];
  else
    z2=R; H=[H;1];
  end
  I=x2-x1; J=y2-y1; K=z2-z1; z3=0; d1=z3-z1; t=d1/K;
  x3=x1+I*t; y3=y1+J*t; V2=[V2;x3,y3,z3];
end
Es=[]; E1=[]; E2=[]; He=[];
for i=1:EN,
  Es{i,1}=[]; E1{i,1}=[]; E2{i,1}=[]; He{i,1}=[];
end
for i=1:EN,
  for j=1:(Res+1),
    x=El{i,1}(j,1);  y=El{i,1}(j,2);  z=El{i,1}(j,3); d=sqrt(x*x+y*y+z*z);
    t=R/d; x1=x*t; y1=y*t; z1=z*t;
    E1{i,1}=[E1{i,1};x1,y1,z1]; x2=0; y2=0;
    if(z1>=0)
      z2=-R; He{i,1}=[He{i,1};0];
    else
      z2=R; He{i,1}=[He{i,1};1];
    end
    I=x2-x1; J=y2-y1; K=z2-z1; z3=0; d1=z3-z1; t=d1/K; x3=x1+I*t;
    y3=y1+J*t; E2{i,1}=[E2{i,1};x3,y3,z3];
  end
end
n=12;
Co=[]; Tmp=[];
for i=0:(pi/Res):pi,
  x=R*cos(i); y=R*sin(i); Co=[Co;x,y,0];
end
CN=size(Co,1); C=[]; count=0; TmpA=[Co,ones(CN,1)];
for t=(pi/n):(pi/n):(pi-pi/n),
  rx=[1,0,0,0; 0,cos(t),-sin(t),0; 0,sin(t),cos(t),0; 0,0,0,1];
  count=count+1; C{count,1}=(rx*TmpA')';
end
Ct=count; C1=[];
for i=1:Ct,
  C1{i,1}=[];
end
for i=1:Ct,
  C1{i,1}=[C1{i,1};TmpA(1,1),TmpA(1,2),TmpA(1,3)];
  for j=2:(CN-1),
    x=C{i,1}(j,1); y=C{i,1}(j,2); z=C{i,1}(j,3); x1=0; y1=0; z1=-R;
    I=x1-x; J=y1-y; K=z1-z; z2=0; d=z2-z; t=d/K; x2=x+I*t; y2=y+J*t;
    C1{i,1}=[C1{i,1};x2,y2,z2];
  end
  C1{i,1}=[C1{i,1};TmpA(CN,1),TmpA(CN,2),TmpA(CN,3)];
end
Co=[];
```

```
for i=(-pi/2):(pi/Res):(pi/2),
  x=R*cos(i); y=R*sin(i); Co=[Co;x,y,0];
end
CN=size(Co,1); count=0;
TmpA=[Co,ones(CN,1)];
for t=(pi/n):(pi/n):(pi-pi/n),
  ry=[cos(t),0,-sin(t),0; 0,1,0,0; sin(t),0,cos(t),0; 0,0,0,1];
  count=count+1; C{(Ct+count),1}=(ry*TmpA’)’;
end
for i=(Ct+1):(Ct+count),
  C1{i,1}=[];
end
for i=(Ct+1):(Ct+count),
  C1{i,1}=[C1{i,1};TmpA(1,1),TmpA(1,2),TmpA(1,3)];
  for j=2:(CN-1),
    x=C{i,1}(j,1); y=C{i,1}(j,2); z=C{i,1}(j,3); x1=0; y1=0; z1=-R; I=x1-x; J=y1-y;
    K=z1-z; z2=0; d=z2-z; t=d/K; x2=x+I*t; y2=y+J*t; C1{i,1}=[C1{i,1};x2,y2,z2];
  end
  C1{i,1}=[C1{i,1};TmpA(CN,1),TmpA(CN,2),TmpA(CN,3)];
end
Ct=Ct+count; Ct=Ct+1; C1{Ct,1}=Co; Tmp=[-Co(:,1),Co(:,2:3)]; Ct=Ct+1; C1{Ct,1}=Tmp;
clf; hold on;
for i=1:Ct,
  for j=1:(CN-1),
    if(mod(j,3)>1)
      tma=[C1{i,1}(j,1),C1{i,1}((j+1),1)];
      tmb=[C1{i,1}(j,2),C1{i,1}((j+1),2)];
      tmc=[C1{i,1}(j,3),C1{i,1}((j+1),3)];
      plot3();
    end
  end
end
axis off; axis equal; clf; hold on; % for below, E2 may be replaced by E1 and El
for i=1:VN,
  if(H(i,1))
    plot3(V2(i,1),V2(i,2),V2(i,3),’o’);
  else
    plot3(V2(i,1),V2(i,2),V2(i,3),’.’);
  end
end
for i=1:EN,
  for j=1:Res,
    if(He{i,1}(j,1))
      tma=[E2{i,1}(j,1),E2{i,1}((j+1),1)];m tmb=[E2{i,1}(j,2),E2{i,1}((j+1),2)];
      tmc=[E2{i,1}(j,3),E2{i,1}((j+1),3)];
      if(mod(j,8)>3)
        plot3(tma,tmb,tmc,’--’);
      end
    else
      plot3(tma,tmb,tmc);
    end
  end
end
for i=(Ct-1):Ct,
  for j=1:(CN-1),
    tma=[C1{i,1}(j,1),C1{i,1}((j+1),1)]; tmb=[C1{i,1}(j,2),C1{i,1}((j+1),2)];
    tmc=[C1{i,1}(j,3),C1{i,1}((j+1),3)]; plot3(tma,tmb,tmc);
  end
end
axis off; axis equal; clf; hold on;
for i=1:EN,
  for j=1:Res,
    if(mod(j,5)>2)
      tma=[El{i,1}(j,1),El{i,1}((j+1),1)];
      tmb=[El{i,1}(j,2),El{i,1}((j+1),2)];
      tmc=[El{i,1}(j,3),El{i,1}((j+1),3)];
      plot3(tma,tmb,tmc);
    end
    if(mod(j,7)>2)
      plot3(tma,tmb,tmc);
    end
    plot3(tma,tmb,tmc);
  end
end
axis off; axis equal; rotate3d;
% for future developments
tx=.2; ty=.1; tz=.3; Vo=V; Tmp=ones(VN,1); V=Vo+[tx*Tmp, ty*Tmp, tz*Tmp];
V1o=V1; V2o=V2; Ho=H; Elo=El; Eso=Es; E1o=E1; E2o=E2; Heo=He;
a=.3; b=.4; c=sqrt(1-a*a-b*b) u=[a,b,c]; t=.5; q=[u*sin(t),cos(t)];
x=q(1,1); y=q(1,2); z=q(1,3); w=q(1,4); tma=[(1-2*(y*y+z*z)),(2*(x*y-w*z)),(2*(x*z+w*y))];
tmb=[(2*(x*y+w*z)),(1-2*(x*x+z*z)),(2*(y*z-w*x))];
tmc=[(2*(x*z-w*y)),(2*(y*z+w*x)),(1-2*(x*x+y*y))]; M=[tma;tmb;tmc]; V=(M*Vo’)’;
```

```
x1=Vo(3,1); y1=Vo(3,2); m=sqrt(x1*x1+y1*y1); c1=x1/m; s1=y1/m; c2=1; s2=0;
t=atan((c2*s1+c1*s2)/(c1*c2+s1*s2));  TmpA=cos(t); TmpB=sin(t); rz=[TmpA,-TmpB,0,0;
TmpB,TmpA,0,0; 0,0,1,1; 0,0,0,1]; hold on;
for i=1:(Ct-2),
  TmpA=[C1{i,1},ones(CN,1)]; Tmp=(rz*TmpA')';
  for j=1:(CN-1),
    if(mod(j,3)>1)
      plot3([Tmp(j,1),Tmp((j+1),1)],[Tmp(j,2),Tmp((j+1),2)],[Tmp(j,3),Tmp((j+1),3)],'-.');
    end
  end
end
```

B

§ B. Terminology

a, A

aboulia. lack of will or initiative seen with organic disease or damage to the brain.

adjacent. have a certain thing in common. ~ **edges**, edges which which have a common vertex. ~ **tiles**, tiles which have a common edge. ~ **vertices**, vertices which have a common edge.

affine. (*Lat.* affinis; *Fr.* affin) relating to a coordinate transformation that is equivalent to a linear transformation followed by a translation. **affine combination**, $p = \sum_1^k \alpha_i p_i$, where $p_1, p_2, \ldots, p_k$ are points in E^d, $\alpha_i = \mathrm{Re}$ and $\sum_{i=1}^k \alpha_i = 1$. ~ **geometry**, studies the properties which are preserved (invariant) under transformations in the affine group where A in $x' = xA + c$ is nonsingular. ~ **hull**, the smallest affine set containing L, where L is a subset of E^d. ~**ly independent points**, $p_i \in E^d, \alpha_i \in \Re, i = 1, 2, \ldots, k$ where $p_2 - p_1, \ldots, p_k - p_1$ linearly independent. ~ **mapping**, is $x' = xA + c$. ~ **set**, a linear combination $p = \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k$ where $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$, an affine set is the translation of a linear set vector subspace or it is simply *flat.*

akinesia. total lack of movement.

alternate. ~ **interior angles**, those angles which lie on the opposite sides of a transversal.

altitude. the line from the vertex of a triangle perpendicular to its opposite side.

anorexia. violent refusal to eat.

antipodal points. those points which admit no parallel supporting lines.

Archimedean polyhedron. a polyhedron whose faces are all regular polygons and whose vertices are all congruent to one another.

arteriosclerosis. A general term for the thickening, hardening, and loss of elasticity of the walls of blood vessels.

atherosclerosis. from *Gr. athero* (gruel, paste) and *sclerosis* (hardness). An arteriosclerosis which is caused by the deposition of materials, for example calcium, cellular waste products, cholesterol, fatty substances and fibrin, on the inner lining of an artery. If occurs at a carotid artery it can cause a stroke, while if at a coronary artery a heart attack.

augmented. (of polyhedra) having one or more k-sided faces replaced by a k-gonal pyramid, cupolar, or rotunda.

automatism. forced obedience to external command.

b, B

bimedian. the line joining the mid points of two opposite sides of a quadrilateral.

block. resistance to movement or thought at any level.

bond. link between two cells which share a face.

boundary. ~ **of a ball**, $\partial N_\epsilon(c) = \{x||x - c| = \epsilon\}$ for $c \in R^m, \epsilon > 0$, a hypersphere in R^m.

bulimia. a violent and insatiable appetite .

c, C

cataclasis. granulation.

cataclasite. rock deformed by shearing and cataclasis.

categorical system. an axiomatic sytem S where each pair of its models is isomorphic with respect to S.

central projection. the one-to-one correspondence between points of the plane $x_{d+1} = 1$ (ie. a space E^d) and points on the hemisphere of S^{d+1} corresponding to $x_{d+1} > 0$.

chain. a planar straight line graph $C = (u_1, u_2, \ldots, u_p)$ with vertex set $\{u_1, u_2, \ldots, u_p\}$ and edge set (u_i, u_{i+1}), $i = 1, 2, \ldots, p - 1$.

-cingulum. *sfx* a belt of 12 triangles.

circumcentre. the centre of a circumscribed circle. ~ **of a triangle**, the point of concurrency of the perpendicular bisectors of the three sides of it.

circumcircle. or circumscribed circle is the circle which contains the three vertices of a triangle and has the circumcentre of that triangle as its centre.

claim. a small theorem to be proved, often it is a theorem presented and proved within the proof of another theorem.

close. ~**d ball**, $\mathcal{N}_\epsilon(c) = \{x||x - c| < \epsilon\}$ for $c \in R^m, \epsilon > 0$. **vertex of** ~ **type**,v where $|T \bigoplus T_1 \bigoplus T_2| = k+2$ if $k = N - 1$, where vertex v is common to three polygons $V(T), V(T_1), V(T_2) \in \mathrm{Vn}_k(S)$.

combinatorial geometry. geometry which characterises the geometrical objects as properties of finite subsets.

complement. $A \backslash B = \{x|x \in A, x \notin B\}$.

completeness. (of an axiomatic system S) impossible to add an independent axiom.

component. connected subgraph.

concurrent. (*Lat.* concurrere) ~ **lines**, three or more line which intersect at the same point (point of concurrency).

congruence transformation. mapping of the Euclidean plane onto itself which preserves all distances.

congruent. (*Lat.* congruentem, *nom* congruens)~ **tilings**,tiles which coincide with each other via a rigid motion of the plane or reflection. ~ **triangle**,a triangle in which a one-to-one correspondence can be established between their vertices such that corresonding sides are congruent and corresponding angles are congruent. Two relevant theorems to the congruence of triangles are the SSS, SAS and ASA theorems.

connected. comprising of only one piece. ~ **graph**, G such that $\forall g, v_i, v_j \in G, \exists v_i g v_j \in G$.

consistency. (of an axiomatic system) making no claims of contradictory statements.

convex. (*Lat.* convexus) ~ **polygon**, the region on the true side of all the half-planes of its sides. Substitution of any point coordinates into all of its half-plane equations yields negative when that point is inside the polygon. ~ **hull**, the smallest convex set containing $P \in E^d$. It defines A and b such that $\forall x \in P, Ax + b \leq 0$. ~ **set**, is a set where a line segment formed by any pair of its points lies within the set.

coplanar points. points which lie on the same plane with each other.

corner. a vertex of a polygon, in order to distinguish itself from that of a tiling.

-corona. *sfx* a crownlike structure of eight triangles. **-mega**~, such a complex of 12 triangles.

corresponding angles. two angles, one exterior the other interior, which lie on the same side of the transversal.

covering. a family of sets which completely covers a plane. ~ **lattice**,a lattice derived from another lattice by exchanging edges with vertices, the position of each edge normally being taken to be that of its mid point.

Coxeter-Todd lattice. the lattice in 12 dimensions which has the maximum packing number.

cross ratio. (of four collinear points p_1, p_2, p_3 and p_4) the ratio $c(p_1, p_2; p_3, p_4) = (p_{13}/p_{14})/(p_{23}/p_{24})$. It is an invariant under a linear transformation.

Curie point. the transition temperature where ferromagnetism changes into paramagnetism. For example, when a piece of iron gets too hot it is no longer attracted to a magnet.

d, D

Delaunay tessellation. see Delone tessellation.

Delone tessellation. decomposes a Euclidean space of m dimensions into simplexes identical with one another through linear transformations.

deltahedra. Polyhedra which has faces all equilateral triangles. There are eight convex deltahedra, namely tetrahedron, octahedron, icosahedron, triangular dipyramid, pentagonal dipyramid, heccaidecadeltahedron, tetracaidecadeltahedron, and dodecadeltahedron.

δ-slice. a portion of E^d, $d \geq 2$ contained between two hyperplanes orthogonal to a coordinate axis and at a distance of 2δ apart.

deltohedron. solids obtained by twisting one cone of the two cones in regular dipyramids by $1/(2n)$ turn. Then the result is called an n-gonal deltohedron and the original polyhedron a regular n-gonal dipyramid.

depth. (*O.E.* deop)(of a point p in a set S) the number of convex hulls or convex layers that have to be stripped from S before p is removed; (of a set S) the depth of its deepest point.

dihedral angle. the angle created by two intersecting planes.

n^p-distribution. distributions whose expected number of extreme points in a sample of size n is $\mathcal{O}(n^p)$.

dominate. (of a point) having coordinate components in all dimensions greater than another.

e, E

ϵ_0. the permittivity of free space, $\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}.

edge. (*Ger.* die Kante, -n; *Lat.* acies) the arc joining two vertices or a 1-face of a d-dimensional polytope P. **~-to-~**, (of a tiling of polygons) having all sides and edges coincides, as well as corners and vertices.

elongated. (of polyhedra) having a largest m-sided polygon replaced by an m-prism.

endpoint. (of edges) a vertex.

equiaffine. a subgroup of affine group whose $|A| = \pm 1$, the invariant of which is the volume; **Euclidean distance.** distance between two points represented as vectors x_i and x_j, that is $|x_i - x_j| = \sqrt{(x_i - x_j)^{\mathrm{T}}(x_i - x_j)} = \left[\sum_{k=1}^{n}(x_{ik} - x_{jk})^2\right]^{1/2}$ for n dimensions.

Euclidean space. a Cartesian space with the Euclidean distance of any dimension.

Euler line. the line containing the circumcentre O, the median point M, the orthocentre H and the centre N of the medial circle. The length of this line is $|OH|$. Then we have $|OM| = (1/3)|OH|$ and $|ON| = |NH|$.

expected complexity. estimate of the average behaviour of an algorithm.

exterior angle. (of a transversal) each of the two angles formed by a halfline, a vertex, and the halfline of a transversal on the side away from the line segment that contains both vertices; (of a triangle) an angle adjacent and supplementary to an [interior] angle of a triangle.

extreme point. (of a convex set) a point $p \in S$ convex set where $\nexists a, b \in S$ such that p lies on the open line segment $\overline{ab}$;

f, F

face. (*Ger.* die Fläche; *Lat.* hedrae).

facet. a $(d-1)$-face of a polytope in d dimensions. see also *subfacet.*

far. (*O.E.* feorr) **vertex of a ~ type**, is when $|T \bigoplus T_1 \bigoplus T_2| = k - 2$, if $k = 1$.

flux. the volumetric flow per cross sectional area, $j = \mathrm{d}V/(A\mathrm{d}t)$.

full period generator. a linear congruential generator whose period is k.

g, G

galactic. *adj* Pertaining or belonging to the Milky Way Galaxy.

galaxian. *adj* Pertaining or belonging to a galaxy.

n-gon. a polygon with n sides and n corners.

gyroelongated. (of polyhedra) having one largest m-sided polygon replaced by an m-gonal antiprism.

h, H

half-plane. (of a plane P with respect to a line m relative to a point $A \in P, A \notin m$) $P_1 = \{A\} \cup \{X | X \in P, X \notin m, (A, X) \cap m = \emptyset\}$ and $P_2 = \{X | X \in P, X \notin m, , (A, X) \cap m \neq \emptyset\}$.

half-space. the portion of E^d lying on one side of a hyperplane. Or if P is a plane and A a point not on P, then the two halfspaces with respect to P relative to A are $\S_1 = \{A\} \cup \{X | X \in P, (A, X) \cap P = \emptyset\}$ and $\S_2 = \{X | X \in P, (A, X) \cap P \neq \emptyset\}$.

n-hedral tiling. a tiling with n distinct prototiles.

homeomorphism. topological equivalence.

homogeneous coordinates. coordinates obtained from projection of points from the inhomogeneous or conventional coordinates represented by the hyperplane $x_{d+1} = 1$ onto the unit hemisphere S^{d+1} of E^{d+1} represents the points at infinity by letting $\varepsilon_{d+1} = 0$.

hyperboulia. excess of will, urgency.

hyperkinesia. increased speed, violence, force, and spread of movement.

hyperplane. a vector space of codimension 1. A hyperplane in an n-dimensional hyperspace is a linear space of $(n-1)$ dimensions. It separates the hyperspace into three parts, *viz.* itself and two other parts which are homeomorphic to the original space.

i, I

incentre. the point where the medians of a triangle intersect.

incidence. the membership of a point p on a line l, is an invariant in affine geometry.

incircle. (of a triangle) the circle which has the incentre of that triangle as its centre and touches all three sides of the triangle.

independence. (of a axiom) cannot be proved by using one or more other axioms within of the same set.

inscribed circle. see *incircle*.

interior angle. (made by a transversal) an angle made by the halfline transversal on the side which contains the segment beween the two vertices, a vertex, and one halfline of the other line attached to that vertex.

inversion. (in E^d) a point-to-point transformation of E^d which maps a vector v applied to the origin to the vector $v' = v\dot{1}/|v|^2$.

isometry. *n* a distance-preserving mapping, a synonym for congruence transformation, comprises of rotation, translation, reflection and the latter two combined which is called a glide reflection; a transformation in which a figure and its image are equal reflections of each other.

isomorphic. one to one. ~ **models**, (in an axiomatic system S) those models in which there exists at least one relation-preserving, one-to-one correspondence between each of their elements, in other words every true statement made about elements in one set is also true about the corresponding elements in the other set.

isomorphism. a one to one and onto relation.

isosceles triangle. a triangle which has two equal sides, *i.e.* where two of the sides are congruent.

isotone. monotone nonincreasing or nondecreasing.

k, K

kagome. *Jpn.* a basket pattern.

l, L

Leech lattice. the lattice in 24 dimensions which has the maximum packing number.

linear. (*Lat.* linearis) ~ **combination**, $p = \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k$ for $p_i \in E^d, \alpha_i \in \Re, i = 1, 2, \ldots, k$. ~ **congruential generator**, a generator which produces random numbers $R_i \in [0, 1)$, where $R_i = X_i / k$, $X_i \in [0, k-1]$, $X_{i+1} = (sX_i + c) \bmod k$, $i = 1, 2, \ldots$, $X_0, m, c, k \in \mathcal{I}^+$, X_0 is the *seed*, m the *multiplier*, c the *increment* and k the *modulus.* ~ **set**, an affine set which passes through the origin.

link. a branch that is not a part of a tree.

lune. a combination where two triangles are attached to opposite sides of a square.

m, M

mapping. association of each preimage point in the domain subset of the source set to exactly one image point in the range subset of the target set. **one-to-one** ~, a mapping where there is no two different images with the same preimage. **onto** ~, a mapping where the range is the union of all subsets of images

medial circle. the circle which passes through the mid points of the sides of a triangle, the feet of its altitudes and the mid of the lines from the point where the altitudes intersect to the corresponding vertices. It is also known as the nine-points circle or the pedal circle because of these nine points it passes through.

median. (of a triangle) bisector of any one of its three angles. also ~ **line**,.

mesh. a closed loop in a graph. **basic** ~, a closed loop formed from the tree by one link of the graph.

metric. a generalized distance.

Möbius strips. An n^{th}-order Möbius strip is a band obtained by joining the two ends of a rectangular strip into a loop after having twisted one of them by an angle $n\pi$, where n is an integer. When the strip is cut along the centre-line, if n is odd the result is one strip having $2n+2$ half twists which is knotted when $n \geq 3$. If n is even the result is two strips.

monohedral tilings. tilings in which every tile congruent with one another.

monotone. (*Fr.* monotonie; *Gr.* monotonos) ~ **chain**, (with respect to a straight line l) a chain which is intersected by a line orthogonal to l exactly one point. ~ **polygon**, (with respect to a straight line l) a simple polygon whose boundary is the union of two chains monotone with respect to l.

mylonite. Laminated rock with fine grain which is the product of grinding or granulation within the tectonic fault zones.

Monte Carlo. method first done during the 1940s, involve partial differential and integral equations and multi-dimensional integrals, stereotypically maps a deteministic system onto a sampling experiment, from which are collected random samples whose results of statistical analysis give an estimate solution to the problem in real system.

n, N

neighbour. that which has a certain thing in common, for example a common vertex or a common edge.

nine-points circle. see medial circle.

o, O

obtuse. (*Lat.* obtusus) ~ **angle**, an angle which is greater than a right angle, or one which is neither a right- nor an acute angle.

open ball. $N_\epsilon(c) = \{x \mid |x - c| < \epsilon\}$ for $c \in R^m, \epsilon > 0$.

order. (*O.Fr.* ordre) ~ **of**, (implies that) multiplication by constants is involved. ~ **of connectedness**, (of a graph) the maximum number of edges which can be removed without changing the number of components. ~ **of a Voronoi diagram**, $k = |T|$, $\mathrm{V}_{n_k}(S) = \bigcup_i V(T)$, $T \subset S$.

orthocentre. (of a triangle) the point where the lines drawn from the vertices normal to the respective oppositesides intersect, the point of concurrency of its three altitude lines.

orthogonal. (*Gr.* orthos; gonia) right angle. ~ **group**, intersection between the similarity and the equiaffine groups, an invariant of which is the distance. ~ **vectors**, those vectors which are at right angle to each other; $|x_1 + x_2| = |x_1 + x_2|$, $(x_1 + x_2)^2 = (x_1 + x_2)^2$, $(x_1 + x_2)^{\mathrm{T}}(x_1 + x_2) = (x_1 - x_2)^{\mathrm{T}}(x_1 - x_2)$, $x_1^{\mathrm{T}} x_2 = 0$. ~ **projection**, a set of orthogonal projections of point.

orthographic. (*Gr.* orthos; graphein; *Lat.* orthographia) ~ **projection**, a set of orthogonal projections of point.

p, P

packing. a family of non-overlapping sets in a plane.

parametral plane. (in crystallography) the plane which cuts all the three axes of a crystal.

partition. (of S) each of the two or more nonempty and disjoint subsets of S.

path. a self-avoiding path, ie a path whose all vertices are distinct.

pencil. (*Lat.* penicillus): ~ **of lines**, a set of concurrent lines.

perseveration. an uncontrollable self-stimulating and self-maintaining which causes the indefinite continuation or repetition of nervous processes.

polyhedral. (*Gr.* polyedros) ~ **set**, the intersection of a finite set of closed half-spaces in E^d.

polytope. (also d-polytope) convex d-polytope or a bounded d-dimensional polyhedral set.

projective. (*Lat.* projectum) ~ **group**, a full linear group on $d + 1$ homogeneous coordinates where $|B| = [A, 0; c, 1] \neq 0$ in $(x', 1) = (x, 1)B$

r, R

regression. a problem of best approximation in a subspace. ~ **function**, some function f^* of $d - 1$ variables which minimizes the norm $| f - f^* |$, where f is a function of $d - 1$ variables which represents a set of points in E^d.

regular. (*Lat.* regula) ~ **polygon**, a polygon with equal sides and equal angles. ~ **polytope**, (or a regular polyhedron) a polytope with all faces congruent regular polygons. There are only five distinct types of these and they are called the Platonic solids; (of a vertex) v_j, when there are $i < j < k$ such that (v_i, v_j) and (v_j, v_k) are edges of a graph G whose vertices are indexed in such a way that $i < j$ means either $y_i < y_j$ or, $y_i = y_j$ and $x_i > x_j$.

remote. (*Lat.* removere, remotus) ~ **exterior angle**, (of an interior angle) an angle not adjacent to the interior angle. ~ **interior angle**, (of an exterior angle) an angle not adjacent to the exterior angle.

rheology. the study of the flow of matter.

ridge. boundary element of a facet.

rigid. (*Lat.* rigere, rigidus) ~ **motions**, affine transformations which preserve distance, which are the essence of Euclidean geometry.

route. a path with possible double points.

s, S

side. edge. A side is to a corner what an edge is to a vertex.

similarity. (*Fr.* similaire; *Lat.* similis) ~ **group**, an affine group which has $AA^{\mathrm{T}} = \lambda^2 I$, the ratio of distances between points are preserved.

simple. (*O.Fr.* simple; *Lat.* simplus) ~ ***d*-polytope**, a polytope whose vertices meet exactly d edges. A simple polytope is a dual of a simplicial one.

simlex. a convex hull. **Euclidean** ~, ***d*-simplex**, a d-polytope P which is the convex hull of $(d-1)$ affinely independent points. It contains the total number of 2^{k+1} of k-faces where $k \in \mathcal{I}\, k \geq -1$, an empty set being $k = -1$. A simplex for $d = 0$ is a vertex, for $d = 1$ an edge, for $d = 2$ a triangle and for $d = 3$ a tetrahedron.

simplicial. (*Lat.* simplex, simplicis) ~ ***d*-polytope**, a polytope all the facets of which are simplices.

simply connected. is connected and contains no holes.

site. a vertex, to be distinguished from a nucleus or generator of Voronoi networks which is also a site but of the dual network.

space. (*O.Fr.* espace; *Lat.* spatium) ~ **points**, at least four points which are not necessarily coplanar.

sparsity. (*Lat.* sparsus) measure of sparseness in point distribution, a point set $S \in E^d$ has sparsity $c \in \mathcal{I} \geq 1$ for a given $\delta \in \mathcal{R}^+$ iff there are at most c points of S within any box or hypercube of side 2δ. In other words, sparsity is the scarcity of points within a given box. It is preserved through orthogonal projection.

spheno-. *prf* a wedgelike combination formed by two adjacent lunes. **di~-**, two such combinations. **hebe~-**, two lunes separated by a third one.

snub. (of polyhedra) *adj* resulting from a chiral process of rotating all faces of a polyhedron in the same direction. This creates one m-sided polygon for each vertex of degree m and two triangles for each edge. A polyhedron has the same snub as its dual.

stereohedra. set of regions whose congruent copies fill three dimensional space without overlap except at their boundaries.

stochastic. (*Gr.* stokhastikos) ~ **systems**, physical systems which involve random process evolving over time.

subfacets. the $(d-2)$-faces of a polytope P in d dimensions.

t, T

temporomandibular jaw joint disease. The condition of painful jaw joint, sometimes also ears, neck, shoulders and back. Possible symptoms include uncomfortability when openning the mouth and clicking sound when moving the jaw joint.

tessellated. (*Lat.* tessera, tessella; *Gr.* tessares) ~ **polyhedra**, polyhedral cells having the same number of faces.

tessellation. (from *Lat. tessellātus.* mosaic) a space comprised of tiles or simplexes identical to one another via linear transformation; space entirely covered with a pattern.

tesseract. a four-dimensional hypercube.

tile. each piece of, or each set in, a tiling.

tiling. (*Lat.* tegere, tegula; *O.E.* tigele) **plane** ~, a countable family of closed sets which covers a plane and leaves no gap.

tinnitus. The perception of clicking, hissing, popping, ringing, rumbling, or other sound in the ears when no external sound is present. Causes are numerous and include bad positioning of the neck and atherosclerosis. Suggested treatment ranges from taking vitamin A, E, magnesium, potassium and zinc to chiropractice.

tree. a set of branches connecting all the nodes of the graph without forming any closed loops or meshes.

truncated. (of polyhedra) having an k-gonal pyramid cut off from one or more of the vertices.

v, V

valence. (*Lat.* valere, valentia) n-~, having a vertex which is an end point of n edges.

vertex. *pl.* vertices. (*Ger.* die Ecke, -n; *Lat.* angulorum solidum) a point in any dimension. When creating a Voronoi diagram the nuclei are vertices of the corresponding Delaunay diagram, while a Voronoi vertex is the circumcentre of a facet of the Delaunay triangulation of a convex hull one dimension higher. ~ **of a pencil**, the point through which all lines in a pencil pass. Vertices are many a vertex or isolated points connected to edges, they are 0-faces of a polytope P in d dimensions.

Voronoi. *n* George Fedosevich Voronoi; Voronoi tessellation, Voronoi networks, *etc.* **generalised ~ diagram**, $V(T) = \{p : \forall v \in T, \forall w \in (S-T), d(p,v) < d(p,w)\}$ or $V(T) = \bigcap_{ij} H(p_i, p_j), p_i \in T, p_j \in (S-T)$, where $H(p_i, p_j)$ is the half-plane containing p_i that which is defined by the perpendicular bisector of $\overline{p_i p_j}$. ~ **diagram**, (also called area of influence polygons (mining), area potentially available to a tree (forestry), capillary domains, Dirichlet tessellation, domain of an atom (metallurgy), plant polygons (ecology), plesiohedra (one kind of stereohedra), Thiessen polygon, Wigner-Seitz regions (physics), Wirkungsbereich (crystallography)).

worst-case complexity. measures the performance of an algorithm over all problem instances.

§ B.1 Abbreviation

a, A

AIESEC. Association Internationale des Etudiants en Sciences Economiques et Commerciales.

ATPIJ. The Association of Thai Professionals in Japan.

ATSIST. The Association of Thai Students in Science and Technology Professions.

c, C

cf. *Lat. confer.*

c.g.. centre of gravity.

d, D

DT. Delaunay triangulation; Dirichlet tessellation.

e, E

eg. *Lat. exempli gratia.*

i, I

IEEE. Institute of Electrical and Electronics Engineers, Inc..

IUPAC. International Union of Pure and Applied Chemistry. `www.iupac.org`.

p, P

p.d.f.. probability density function.

s, S

s.t.. such that.

t, T

TC. *acronym* Twentieth Century.

TIT. *acronym* Tokyo Institute of Technology.

u, U

UMIST. *acronym* University of Manchester Institute of Science and Technology.

v, V

VP. *acronym* Voronoi Percolation; the study of percolation on Voronoi networks; any application of Voronoi tessellation in the percolation theory.

VT. Voronoi tessellation.

C

§ C. Resources

§ C.1 Biographies

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi. *b. circa 780, Baghdad; d. circa 850.*

Lidwig van Beethoven. *b. 17th December 1770 , Bonn; d. 26th March 1827 , Vienna. An accomplished pianist and composer who wrote most wonderful piano sonatas and helped shape the Romantic period of classical music. He began to have problems with his hearing facility in 1796 which developed to become a total deafness which most modern otologists decide are caused by otosclerosis of the mixed type.*

Brahmagupta. *b. 598, Ujjain, India; d. circa 670, India.*

Jean Le Rond d'Alembert. *b. 17th November 1717 , Paris, France; d. 29th October 1783 , Paris, France.*

Girolamo Cardano. *b. 24th September 1501 , Pavia, Duchy of Milan; d. 21st September 1576 , Rome.*

Augustin Louis Cauchy. *b. 21st August 1789 , Paris, France; d. 23rd May 1857 , Sceaux, France.*

Arthur Cayley. *b. 16th August 1821 , Richmond, Surrey, England; d. 26th January 1895 , Cambridge, Cambridgeshire, England. Another mathematician from a Yorkshire family who lived in Cambridge. He contributed to matrices, non-Euclidean geometry and the abstract group concept.*

Edward Salisbury Dana. *b. 16th November 1849 , New Haven, Connecticut; d. 1935. Studied at Yale, where he received his Ph.D. in 1876, as well as in Heidelberg and Vienna, he was the son of James Dwight Dana who also wrote books as well as appendices to his father's System of Mineralogy.*

James Dwight Dana. *b. 12th February 1813 , Utica, New York; d. 4th April 1895 . Studied at Yale and joined the navy, he taught at Yale and married Henrietta Frances, the third daughter of Professor Benjamin Silliman whom he assisted there. He wrote a most definitive Manual of Geology, received a Ph.D. from the University of Munich on its fourth centennial celebration in 1872, and continued working very hard into the last year of his life.*

Henry Philibert Gaspard Darcy. *b. 30th June 1803 , Dijon, Départment de la Côte d'Or, France; d. 2nd January 1858 , Paris. Discovered the Darcy's law of flow in porous media. Invented* *the modern style Pitot tube. Noticed the existence of the boundary layer in fluid flow. His name is sometimes wrongly written 'D'Arcy'. This has been verified as his school photo in 1821 already wrote the name as 'Darcy'. It is interesting that his name should have always been written in such an anglicised way as 'Henry', whereas the name of his wife, on the other hand who was originally English, used a french spelling, 'Henriette Carey'.*

Jean Baptiste Louis Romé *Delisle. b. 1736, Gray, eastern France; d. 1790, Paris.*

Boris Nikolaevich Delone. *b. 15th March 1890 , St Petersburg, Russia; d. 1980.*

René *Descartes. b. 31st March 1596 , La Haye, Touraine, France; d. 11th February 1650 , Stockholm, Sweden. In 1647 he met Pascal in France and argued with him that a vacuum could not exist. The curve of the equation* $x^3 + y^3 = 3axy$ *which he discussed in 1638 is now called the Folium of Descartes, though it is no longer associated with flowers' petals. He solved* $x^2 + ax = b^2$ *with ruler and compass by writing it as* $(x + a/2)^2 = (a/2)^2 + b^2$ *and seeing that* x *is nothing but the distance from the corner* A *to the circle, centred at* B*, which has a diameter of* a *and touches a line segment of length* b, $\overline{AC}$, *at* C.

Abraham de Moivre. *b. 26th May 1667 , Vitry near Paris, France; d. 27th November 1754 , London, England.*

Johann Peter Gustav Lejeune Dirichlet. *b. 13th February 1805 , Düren, French Empire; d. 5th May 1859 , Göttingen, Hanover. The young from Richelet or Le jeune de Richelet, for the town in Belgium where his family came from, he is not from France as many had claimed. In his youth he showed interests in history and mathematics. He treasured his copy of Gauss's Disquisitiones arithmeticae as others might a bible. When Gauss died in 1855, he was offered his chair at Göttingen. With him the golden age of mathematics in Berlin began. His proofs are characterised by surprisingly simple initial observations followed by extremely sharp analysis of the problem.*

Diophantus of Alexandria. *b. circa 200; d. circa 284.*

Johann Peter Gustav Lejeune Dirichlet. *His name originated from Le jeune de Richelet [The young from Richelet, a town in Belgium where his family came from.]*

Euclid of Alexandria. *b. circa 325 BC; d. circa 265 BC, Alexandria, Egypt.*

Leonhard Euler. *b. 15th April 1707 , Basel, Switzerland; d. 18th September 1783 , St. Petersburg, Russia.*

Edward Morgan Forster. *b. 1879; d. 1970.*

Johann Carl Friedrich Gauss. *b. 30th April 1777 , Brunswick, Duchy of Brunswick; d. 23rd February 1855 , Göttingen, Hanover. At seven he discovered that* $\sum_{i=1}^{1} 00i = 50 \times 101$. *His doctoral dissertation was a discussion of the fundamental theorem of algebra. He is interested in differential geometry, where he discovered that the Gaussian curvature is invariant under isometric transformations of area in* E^3. *He is also interested in magnetism and worked with Weber. Together they discovered the Kirchoff's theory.*

William Rowan Hamilton. *b.* 3rd *or* 4th *August, 1805; d. 1865. When a child, he was taught 14 languages, and at 17 taught himself mathematics and there by discovered an error in Laplace's Celestial Mechanics. He is credited for having invented the quaternions, and sometimes for having scratched the result of that discovery, that is* $i^2 = j^2 = k^2 = ijk = 1$, *on the stone of the Brougham bridge on the Royal Canal. He also invented the icosian game where one is asked to find a path along a polyhedron's edges such that each node is visitted only and at least once.*

Abraham bar Hiyya Ha-Nasi. *b. 1070, Barcelona, Spain; d. 1136, Provence, France.*

Henry Selby Hele-Shaw. *b. 1854, Billericay, Essex; d. 1941. He taught at University College, Liverpool. He was elected to the Royal Society in 1899 because of the fundamental investigation he had carried out regarding streamline flow of liquids.*

Charles Hermite. *b. 24th December 1822 , Dieuze, Lorraine, France; d. 14th January 1901 , Paris, France.*

Joseph-Louis Lagrange. *b. 25th January 1736 , Turin, Sardinia-Piedmont; d. 10th April 1813 , Paris, France.*

Pierre Laplace. *b. 1749; d. 1827. It is often said that his five volume Mécanique Céleste (1799–1825) is in great part a summation of works by his predecessors, as a result of which he often omitted derivations by writing them off as being obvious and easy to see.*

Sir Joseph Larmor. *b. 11th July 1857 , Magheragall, County Antrim, Ireland; d. 19th May 1942 , Holywood, County Down, Ireland. He was a Lucasian Professor of Mathematics at Cambridge from 1903 until 1932 when he was succeeded by Dirac.*

Gottfried Wilhelm von Leibniz. *b. 1st July 1646 , Leipzig, Saxony; d. 14th November 1716 , Hannover, Hanover.*

Hendrik Antoon Lorentz. *b. 18th July 1853 , Arnhem, Netherlands; d. 4th February 1928 Haarlem, Netherlands.*

William Hallowes Miller. *b. 1801, Velinde, near Llandovery, South Wales; d. 1880, Cambridge.*

Hermann Minkowski. *b. 22nd June 1864 , Alexotas, Russian Empire, now Kaunas, Lithuania; d. 12th January 1909 Göttingen, Germany.*

Franz Ernst Neumann. *b. 1798, Joachimsthal; d. 1895.*

Luca Pacioli. *b. 1445, Sansepolcro, Italy; d. 1517, Sansepolcro, Italy. His Summa de arithmetica geometria, proportioni et proportionalita, published in 1494, summarises the contemporary algebra, arithmetic, geometry, and trigonometry. In 1509 published the Divina proportione which deals with the golden ratio and contains illustrations by Leonardo da Vinci.*

Parmenides. *b. circa 515; d. after 450 BC. He reasoned that since a void is nothingness, if two particles were separated by a void, then they would be separated by nothing. In other words, they would not be separated at all, they would be touching (cfDavies, 2001).*

Jean Louis Poisseuille. *b. 1799; d. 1869.*

Simeon-Denis Poisson. *b. 1781; d. 1840.*

Scipione del Ferro. *b. 6th February 1465 , Bologna, Italy; d. 5th November 1526 , Bologna, Italy.*

Niels Stensen (aka Nicolaus Steno). *b. 1638, Copenhagen; d. 1686, Schwerin.*

Robert Louis Stevenson. *(1850–1894)*

James Stirling. *b. May 1692, Garden near Stirling, Scotland; d. 5th December 1770 , Edinburgh, Scotland.*

Johannes Diderik van der Waals. *b. 23rd November 1837 , Leyden, The Netherlands; d. 8th March 1923 , Amsterdam. It was him who coined the equation for real gas,* $(p + an^2/V^2)(V - nb) = nRT$.

George Fedosevich Voronoi. *b. 28th April 1868 , Zhuravka, Poltava guberniya, Russia (now Ukraine); d. 20th November 1908 , Warsaw, Poland. Both his master's degree, 1894, on the algebraic integers associated with the roots of an irreducible cubic equation and his doctoral thesis on algorithms for continued fractions were awarded the Bunyakovsky prize by the St. Petersburg Academy of Sciences. But he decided that he wanted to teach at the Warsaw University where he extended work by Zolotarev on algebraic numbers and the geometry of numbers. He met Minkowski in 1904 at an international conference at Heidelberg.*

Egor Ivanovich Zolotarev. *b. 12th April 1847 , St Petersburg, Russia; d. 19th July 1878 , St Petersburg, Russia. Received a silver medal from the Gymnasium in St. Petersburg, attended lectures by Kummer and Weierstrass, and discussed mathematics with Hermite, he worked with Korkin and gives complete solutions to the four- and five variable cases of the problem of finding the minimal values of* n*-variable quadratic forms with real coefficient.*

§ C.2 Computation and softwares

AVS. *AVS can be used to find cross sections in 2– and 3–D networks.*

hull. *Hull is written in ANSI C by Ken Clarkson. It computes the convex hull of a point set of any dimension.*

Synopsis: hull -d -f<format> -A -aa<alpha> -af<format> -oN -ov -s<seed> -r -m<multiplier> -X<debug file> -i<input file> -oF<output file>
Hull takes in points as its input. The outputs are vertices of the convex hull facets, Delaunay triangulations, alpha shapes, and volumes; in postcript or OFF format for geomview.

hullio.a precursor to *hull.*

Matlab. *Matlab has an algorithm for finding 2–D voronoi diagrams. However, the output data for this is not very structured, which makes it difficult to use the data obtained for analysis purpose.*

When repeatedly running a .m file online, all variables should be cleared by 'clear all' two times, both at the beginning and at the end of the file. Failing to do so sometimes results in consistency in the results.

Submitting .m files through NQS often requires writing every path in full. However, in .m files one can write 'path(path,*directory*)' or 'path(*directory*, path)' for post-appending and pre-appending a path, for example that which contains the .m files containing functions.

I guess that everything done on a matrix in Matlab is as fully vectorised as possible, since provided that this is the case that very program can still be greatly improved, and I hold the programmers who develop it in a better regard than what would have allowed me to assume this. So we can vectorise our algorithm by simply putting the various items into a matrix and work on them, in that matrix, in parallel instead of in sequence as we would normally do. There is still a limitation in that we can subject a matrix to only one operation, and therefore can only do in parallel things which require the same operation.

Another limitation is that we can only put matrices into a structure, which in Matlab contains data of the class *cell*, but not vice versa. So the great convenience we have from working with the cell structure comes at the cost to parallelisation. In effect, this means that we can only parallelise our algorithmic details but not the whole algorithm itself. In other words, we can not simultaneously find the percolation probability for two different networks using Matlab.

qhull. *A quick hull C program for finding convex hulls, Delaunay triangulations, Voronoi vertices, furthest-site Voronoi vertices, halfspace intersection about a point, hull volume and facet area. It is written by C. Bradford Barber and Hannu Huhdanpaa at The Geometry Center, University of Minnesota (Barber et al, 1996). The program combines the 2-d Quickhull algorithm with the general dimension Beneath-Beyond algorithm. The latter is an incremental algorithm which adds a point to the convex hull of the points just processed. It processes a new point in steps as described in Algorithm 10.1. The boundary of the visible facets is the set of horizontal ridges for the point. A facet is visible to a point if the latter is above it. Cones of new facets are constructed from the point of its horizon ridges.*

Algorithm 10.1 *Beneath-Beyond algorithm*

```
for [each new point] do
    locate facets visible to it;
    construct a cone of new facets;
    delete the visible facets;
endfor
```

□

There are various kinds of quickhull algorithms. The one mentioned here is the work of Barber *et al* (*ibid.*) which works in the space of points and convex hulls and maintains an outside set for each facet. Being in the outside set implies that a point is above the facet. It is one of the variations of the randomised incremental algorithm proposed by Clarkson and Shor (1989) which works in the space of halfspaces and polytopes, dual to that of the present one, and maintains a conflict graph, a set of all the list of polytope edges that intersect an unprocessed halfspace. Some of the options are *d* Delaunay triangulation by lifting points to a paraboloid,

d Qu computes the furthest-site Delaunay triangulation from the upper convex hull, *f* print all fields of all facets, *FA* computes total area and volume, *FN* lists the voronoi vertices for each voronoi region, *Fp* halfspace intersection coordinates (*F* for output format), *Ft* prints a triangulation (with points (the centrums) added to non-simplicial facets), *Fv* when used with *v* option prints a (furthest-site) Voronoi diagram (Output: number of ridges \\ `< the count of indices> <1st input site> <2nd input site> <1st ridge> <2nd ridge> ... \\ ...`), *Fx* convex hull vertices, *G* Geomview output (2– to 4–*d*), *H*n,n,... computes halfspace intersection about $[n, n, 0, \ldots]$ (The point $[n, n, n, \ldots]$ lies inside $Hx + b \leq 0$, default $b = 0$), *m* Mathematica output (2– and 3–*d*, in Mathematica <*variable*>, <<, <*filename*> then *Show[Graphics3D[list]]*), *o* prints the input points and facets, *Pg* prints only good facets, *QV*n a good facet includes point n ($n < 0$ means a good facet does not include point n), *Qg* builds good facets, *TO ¡file¿*, *Tv* verify structure, convexity, and point inclusion of the result, *p* vertex coordinates, *i* vertices incident to each facet, *P* printing, *Q* qhull control, *Qb*k:n scales the k^{th} coordinate of the input points (the lower bound of the input points becomes n) *Qb*k*:0B*k*:0* drops dimension k from the input points before the Delaunay and Voronoi to allows sub-dimensional convex hulls, *QB*k:n the upper bound becomes n, *Qbb* scales the last coordinate to $[0, m]$ where m is the maximum absolute value of the other coordinates, *QbB* scales the input points to fit the unit cube after projection to the paraboloid. The lower and the upper bounds for all dimensions are -0.5 and $+0.5$, *QJ* triangular output, *Fa* prints area for each facet, *PA*n prints the n largest facets, *PF*n prints facets larger than n, *Fs* prints a summary of the structure, *s* prints a summary to stderr, *T* tracing, *v* Voronoi diagram via the Delaunay triangulation, *v Qu* finds the furthest-site Voronoi diagram,

Examples:

```
rbox <number of points> t<seed> D3 | qhull QV5 v p Pg | qhull G > <filename>;
rbox <number of points> t<seed> D3 | qhull v Fs > <filename>;
```

A *d*-d convex hull in this algorithm is represented by its vertices and $(d-1)$-d facets or faces. Extreme points are those which are the vertices of a convex hull. Each facet has a set of vertices, a set of neighbouring facets and a hyperplane equation. A $(d-2)$-d face is a ridge of the convex hull, which is the intersection of the vertices of two neighbouring facets. An oriented hyperplane through *d* points is represented by its unit normal which points outwards and its offset from the origin. The signed distance from a point to a hyperplane is the inner product of the point and the normal plus the offset. A hyperplane defines a halfspace of points having negative distance from it. The point is above the hyperplane if this distance is positive.

In R^d, Quickhull repairs in order faults where more than two facets meet at a ridge, a facet is in another facet, a facet has fewer than d neighbours, a facet has a flipped orientation, a point just processed is coplanar with a horizontal facet, concave facets, coplanar facets and redundant vertices.

The program *rbox* is written in C. It generates pseudo random points for *qhull*. The arguments *t*n tells it to use n as a random seed. Here *D2* means 2-dimension while *D3* 3-dimension.

rbox. *This program generates random points for qhull. When used without an option gives a list of possible options.*

Sweep2. *This is a program for creating 2-dimensional Voronoi diagrams and Delaunay triangulation. It uses sweepline algorithm and is written by Steve Fortune.*

triangle. *A C program by Jonathan Richard Shewchuk for 2–D triangulation and mesh generation. It uses Ruppert's Delaunay refinement algorithm.*

volume. *Written by Joseph O'Rourke, it finds the volume of a simple polyhedron from the triangulated surface input read from stdin. Inputs are vertex coordinates represented as integers and triangle faces as vertex indices.*

§ C.3 Internet resources

`www.faqs.org` *Internet FAQ Consortium.* A site for Frequently Asked Questions which covers various areas. Useful algorithms for geometrical computation can be found here together with their references.

`www.geom.umn.edu` *Geometry Center.* Being at the University of Minnesota, this is the place to find the programs Quickhull (Qhull) and Rbox used in the beginning of this study. Qhull is incorporated into MATLAB as the commands Qhull, Convhulln, Delaunayn and Delaunay3. It is written by C. Bradford Barber and Hannu Huhdanpaa.

`www.gnu.org` *The "GNU's not Unix!" Project and Free Software Foundation.* Recursively named, this project originated by Richard Stallman offers a variety of profound softwares, most of which are for Unix and Linux developed by tinkerers. Most of the softwares come with codes, therefore are ideal for developpers. The 3-d viewing program called Geomview is only one example.

`www-groups.dcs.st-and.ac.uk` *Turnbull Server.* Named after Herbert Westren Turnbull (1885–1961), this server is valuable to anyone who has an interest in Mathematics, as well as researchers in history of this field. It houses the MacTutor History of Mathematics archive, which covers history of the subject and biographies of mathematicians. The materials offered are extensive.

`www.gutenberg.org` *Gutenberg Project.* A site containing valuable books on many topics, including but not only literatures.

`www.mathworks.co.uk` *MathWorks Developers of MATLAB and Simulink.* A useful site for users of both products. One of the strong points of products of MathWorks is the extensiveness of their help facilities. The helps available here are better organised and more explanatory than those that come with the program.

`www.nectec.or.th` *National Electronics and Computer Technology Center, Thailand.* Supposedly the only research authority in Thailand, this site hosts various other sites of thai researchers all over the world. That of ATSIST, www.nectec.or.th/bureaux/atsist, is but one of them.

D

§ D. Bibliography

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§ D.1 My writings, Kittisak Nui Tiyapan

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§ E. Essays and relevant publications

§ E.1 Antimonytrioxide extraction from ore by hydrometallurgy

Antimonytrioxide extraction from stibnum ore by hydrometallurgical method
written in, and translated from Thai by Kittisak Nui Tiyapan†
1st March 1991

The importance and the original of the problem

Because Thailand has a large quantity of stibnum ores. But these are mostly of low grade, which makes it unsuitable to extract them by using heat (pyrometallurgy). The study of chemical methods of extraction will be beneficial because, apart from being economical, they better facilitate the control of the various variables involved. Also nowaday there is a continuous increase in the amount of Sb_2O_3 in industry. Preparation of Sb_2O_3 from stibnum metal has quite a high cost. Therefore there need to be a research towards methods of preparing Sb_2O_3 directly from from the stibnum ore which will help reduce greatly both the number of processing steps as well as the cost of production.

Purpose

The purpose of this experimental work is in order to find a suitable condition for the preparation of Antimonytrioxide directly from stibnum ores with the use of a hydrometallurgical process, by investigating the influence of those variables which affect the solubility, namely the temperature and time of leaching, the concentration of the leaching reagent used, and the concentration of the solution.

Abstract

Antimonytrioxide (Sb_2O_3) can be directly and efficiently prepared from antimony ores by the hydrometallurgical process one method of which is acid leaching. By studying the influence of the various variables which effect the efficiency of leaching, the sizes, and the shapes of of Sb_2O_3 crystals, it has been found that a suitable condition for the preparation of Antimonytrioxide from a stibnum ore which contains approximately 35.3% of Sb is by leaching it in the hydrochloric acid solution of ferric chloride at the temperature around 70–90°C requires approximately 1–1.5 hours by using $FeCl_3 \cdot 6H_2O$ approximately 10–20% in excess.

1. Introduction

Antimonytrioxide (Sb_2O_3) is one of the major compounds of antimony. It is in the form of white crystals, has two types of structure, namely cubic (Senamonite) and rhombic (Valentinite), the molecular weight 291.5, the melting point 656°C, the boiling point 1,425°C, the specific gravity 5.2, and the hardness 2.0–2.5 Mohs scale. The solubility in water is 0.01 g/l at 100°C, slightly soluble in basic solution with increasing solubility as the basicity and the temperature of the solvent increase, highly soluble in the hydrochloric and the sulphuric acid, insoluble in the nitric acid, highly soluble in sulfide where the result is the thioantimonate solution, and extremely high solubility in the tartaric acid where it results in the antimony tartate solution.

Usage of antimony trioxide

1. Used as opacifier and emulsifier in enamel production. Being at the same time a good opacifier and a good emulsifier, Sb_2O_3 makes quality enamel that gives an enduring and glossy surface.
2. Used as pigment and lacquer. Sb_2O_3 makes good white pigment which has low rate of oil absorption (approximately 11–13 g / 100 g pigment), can withstand chemicals, light and heat.

Because Sb_2O_3 has a crystalline structure, it helps make strong pigment films. But paints which have Sb_2O_3 as pigment will dry slowly and soft, therefore it is often used together with ZnO in order to make pigments which dry quicker, stronger and have high adhesive strength.

Furthermore, Sb_2O_3 is a fireproof material that is waterproof, rustproof and insulate against heat, which makes it popular in making paints for ships, buildings and cables.

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† c/o Minineral Engineering Department, Chulalongkorn University, Bangkok, Thailand.

3. Used in glass and ceramic industries. Sb_2O_3 is used as decolouriser and clearifier in the manufacturing of glass. The green colour of FeO in glass normally affects its colour, glossiness and transparency . But if Sb_2O_3 is added while the glass is melted, it will sink down in the middle, and upon being heated will act as a decolouriser by reacting with FeO. Apart from this, at a very high temperature Sb_2O_3 will undergo a reduction reaction by releasing O_2. At the same time, it can be volatile at high temperature. The release of O_2 and volatile gas by Sb_2O_3 helps eliminating air bubbles produced while the glass melts, and gives a more transparent glass, that is it acts as a clearifier.
4. Used as flame retardants in fibres and plastics. Flame retardants in general have the following properties.
    - 4.1 Reduce the combustion of fibres,
    - 4.2 prevent the spreading of flame,
    - 4.3 reduce the effect of heat on combustion,
    - 4.4 prevent smoke creation.

An ideal flame retardant when heated should produce a substance which can destroy the combustion mechanism. This function is called *synergesgic system.* For example, Sb_2O_3 in halogen polymer after heating fibres which are treated with Sb_2O_3 there will be chloride smoke produced which will reduce the air for the flame and finally extinguish it.

Sb_2O_3

in certain kinds of plastic not only helps reduce the ability to burn, but also help increase the resistance to heat, light and air.

Also, in the United States Sb_2O_3 has been widely used as flame retardants in wall papers and kitchen utensils. But in the case of flame retardant in carpets, hydrated aluminium oxide has more advantages because Sb_2O_3 when used in combination with some of the fillers in carpets will release toxic gases.

5. Other usages.
    - 5.1 Used in combination with antimony sulphide as filler in rubber industry,
    - 5.2 used as precipitant in manufacturing titanium white,
    - 5.3 added in petrol to reduce pollution,
    - 5.4 used as catalysts in synthetic processes,
    - 5.5 *etc.*

From the statistics of usage of antimony compounds in industry we find that antimony trioxide is the most useful, which is used mainly as flame retardant. We find that during the past 5 to 10 years the amount of Sb_2O_3 used as flame retardant in various kinds of material in Japan and the United States has rapidly increased, as can be seen in the table which forecast the amount of Sb_2O_3 compounds used in 2000 for the US.

Usage	**1984**	**1985**	**1986**	**1987**	**1988**
flame retardants	8,181	7,409	7,961	8,122	8,796
glass	311	298	183	199	294
paint/pigment	275	137	107	97	73
export (estimate)	21	24	24	32	75
others	410	279	212	126	130
total	9,198	8,147	8,687	8,576	9,368

Table 1 *Usage of* Sb_2O_3 *in Japan (tonnes).* **source:** *statistics of Nissho Iwai Co. Japan.*

Country	**1984**	**1985**	**1986**	**1987**	**1988**
S. Korea	–	–	10	69	44
China	1,076	1,103	1,950	2,194	4,066
Taiwan	–	36	36	51	139
Hong Kong	17	34	35	–	24
France	99	112	252	300	258
Belgium	2	–	18	24	54
Italy	–	–	–	–	–
U.K.	1,000	1,212	1,117	1,578	1,359
USSR	241	160	80	87	24
USA	2	15	91	35	108
Mexico	30	–	20	28	72
Bolivia	–	–	–	–	–
S. Africa	–	–	–	2	–
total	2,467	2,672	3,609	4,368	6,188

Table 2 *Antimony trioxide import in Japan (tonnes).* **source:** *statistics of Nissho Iwai Co. Japan.*

			2000		
			congingency forecasts for USA		
End use	1983	statistical projections	Forecast range		probable
			Low	High	
transportation	11,500	23,000	5,000	26,000	13,000
flame retardants	10,000	23,700	10,000	25,000	20,000
rubber products	1,000	3,900	500	1,500	1,000
chemicals	2,000	7,600	2,500	5,000	3,000
ceramics and glass	1,800	2,800	1,500	3,000	2,500
machinery	1,800	1,000	1,000	2,500	1,500
other	1,612	0	1,500	3,000	2,000
total	29,712	–	22,000	66,000	43,000

Table 3 *Predicted usage of antimony trioxide in 2000 of the United States (tonnes).* **source:** *statistical projections, provided by the branch of Economic Analysis are derived from regression analyses based on 24 year historical time series data and from forecasts of economic indicators such as GNP and FRB index. A statistical projection of zero indicated that demand will vanish at or before 2000, based on the historical relationship*

Standard for buying and selling of antimony trioxide

1. ASTM has set the standard for commercial grades of Sb_2O_3 as follows. Sb_2O_3 content, 99.2 – 99.5 %; Impurities (As, Fe, Pb, *etc*), ¡0.5 %
2. JIS has set the standard (JIS K 8407) for commercial grades of Sb_2O_3 as the following.

$Sb_2)_3$	Pb	Fe	Cl	SO_4
$\geq$ 98.0 %	$\leq$ 0.02 %	$\leq$ 0.01 %	$\leq$ 0.1 %	$\leq$ 0.01 %

3. Others. For Sb_2O_3 of other grades, the settig of standards depends on the usages, for example the standard set by Nihon Seiko Co.

3.1 Thpe PATOX-U: Ultra fine particle antimony trioxide. Because this is used in specialised catalyst application which requires high surface activity, the standard is set as sizes of particle at 0.01–0.02 μ, which gives surface area as high as 50–100 m^2/g.

3.2 Thpe PATOX-L:Course particle antimony trioxide. Because this is used in reactive applications, the standard set for the purity is the following.

$Sb_2)_3$	Pb	Fe	SO_4	H_2O
$\geq$ 99.3 %	$\leq$ 0.03 %	$\leq$ 0.003 %	$\leq$ %0.005	$\leq$ 0.1%

Antimony trioxide manufacturing technology

Antimony trioxide can be prepared from the antimony ore by using high temperature or by chemical leaching, which can be briefly described as the following.

1. By high temperature (pyrometrllurgical process)

In this process we must first extract high concentration antimony metal, then pass it through the oxidation process to obtain the white residue of antimony trioxide, as shown in Figure 1. This process is rather complicated and the cost quite high.

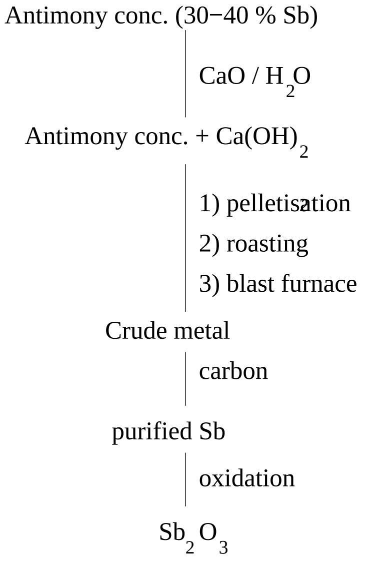


Figure 1. Flow chart showing the process of manufacturing antimony trioxide by pyrometallurgy.

2. By chemical leaching (hydrometallrgical process) In this process we can directly prepare antimony trioxide from the ore. By using the principles of leaching and precipitation, we can easily eliminate impurities, which means a reduction in complexity and saving of energy. On top of that, we may also recycle the leaching agent.

   The process of hydrometallurgy can be divided into two stages, namely

   2.1 Alkali leaching. For example,

$$\begin{array}{ccc} Sb_2S_3 + 3Na_2S & \xrightarrow{\hspace{2em}} & 2Na_3SbS_3 \\ Na_3SbS_3 & \xrightarrow{\text{electrolysis}} & Sb \\ Sb & \xrightarrow{\text{oxidation}} & Sb_2O_3 \end{array}$$

   2.2 Acid leaching. For example,

$$\begin{array}{ccc} Sb_2S_3 + 6FeCl_3 & \xrightarrow{\text{HCl}} & 2SbCl_3 + 6FeCl_2 + 3S^0 \\ 4SbCl_3 + 5H_2O & \longrightarrow & Sb_4O_5Cl_2 + 10HCl \\ Sb_4O_5Cl_2 + NH_4OH & \longrightarrow & 2Sb_2O + 3 + NH_4Cl + HCl \end{array}$$

Not only is the acid leaching more convenient than the alkali leaching, but we can also recycle the leaching agent ($FeCl_3$) for reuse by passing the solution $FeCl_2$ through air or oxygen gas, and the product $FeCl_3$ is easily obtained. Therefore this process merits a study in greater details.

The chemical extraction of ore not only is not complicated, requires lower cost and energy than extraction by heat, but can also reduce the problems of the effects on the environment that stem from the extraction by heat. Furthermore, this chemical extraction is also suitable to complex antimony ores with high impurities and difficult to extract by usual process, for example Jamesonite which is found in great quantity in the northern part of Thailand.

Because nowaday the demand of antimony trioxide in industry has greatly increased, it becomes more necessary to utilise complex- and low grade antimony ores. There is also the need to prevent the effects on the environment. Therefore chemical extraction of antimony, especially the process of acid leaching, is a technology of considerable importance both in the present and in the future.

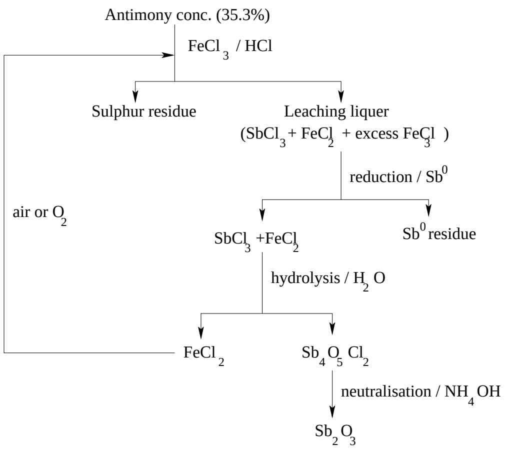


Figure 2

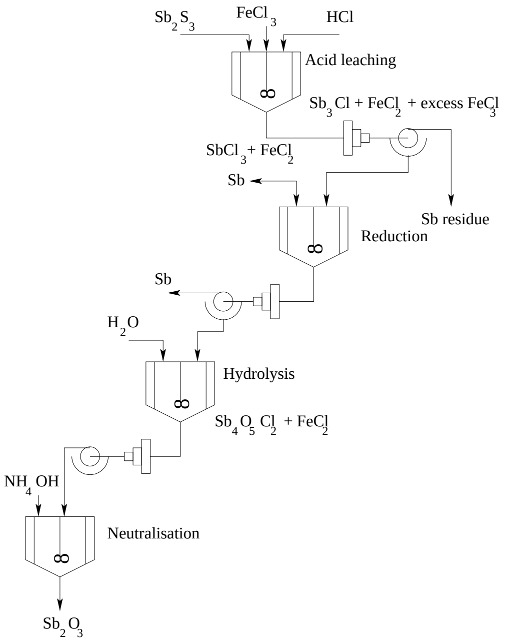


Figure 3

Figure 2 *Flow diagram showing the process of antimony trioxide extraction.*

Figure 3 *Flow diagram modelling the* Sb_2O_3 *extraction in industry.*

2. Experiments

Details of the experiments

1. Learn the methods and techniques of the processing of stibnum ore by the method of hydrometallurgy.

2. Analyse the composition of the stibnum ore sample that will be used in the analysis, in order to be able to find the suitable condition.
3. Find the suitable condition for the preparation of antimony trioxide that is in accordance with the industrial standards by studying the influence the variables have on the solubility efficiency, while maintaining high efficiency of production.

3.1 Study the effect of the temperature of solution
3.2 Study the effect of the time of solution
3.3 Study the effect of the concentration of the acid used in the solution
3.4 Study the effect of the concentration of the solution

Experimental method

1. Weigh 14.29 g of the finely crushed antimony ore (−100#).
2. Dissolve 38.93 g $FeCl_3 \cdot 6H_2O$ in 7N HCl and then heat by using heating mantel until the desired temperature is reached.
3. Slowly add the weighed ore into the solution and start timing.
4. Use pipette to extract the solution at 15, 30, 45, 60 and 120 minute to assay for the amount of Sb extracted at these times.

Assaying for %Sb at each condition

1. Use pipette to extract 2ml of the extracted solution at each condition, add 100ml distilled water in order for $SbCl_3$ to precipitate into $Sb_4O_5Cl_2$.
2. Filter the precipitate using number 42 filtering paper.
3. Dissolve the precipitate in concentrated HCl.
4. Add 1 g tartaric acid.
5. Drop a piece of red litmus into the solution, add 50ml water.
6. Slowly turn the solution into an alkaline solution by gradually adding NaOH until the litmus paper turns from red into blue.
7. Slowly drop concentrated HCl until the solution changes back to become acidic.
8. Add 1 g $NaHCO_3$ to turn the solution alkaline.
9. Add starch solution as an indicator.
10. Titrate with 0.1 N iodine solution.

Method for Sb_2O_3 synthesis

1. Dissolve 14.29 g finely crushed ore into 7N HCl in which 38.93 g of $FeCl_3 \cdot 6H_2O$ has been dissolved.
2. Filter out the insoluble parts from the solution extraction.
3. Add finely crushed Sb metal into the solution thus filtered in order to turn all the excess $FeCl_3$ into $FeCl_2$ by stirring at room temperature for approximately 2 hours.
4. Filter out the insoluble parts.
5. Dilute the solution to 0.55N HCl concentration as a hydrolysis to precipitate $Sb_4O_5Cl_2$.
6. Adjust the pH of the $Sb_4O_5Cl_2$ precipitate to 7 by using NH_4OH solution at 70°C as the neutralisation. The result is the Sb_2O_3 desired.
7. Filter and then bake dry.

3. Experimental results

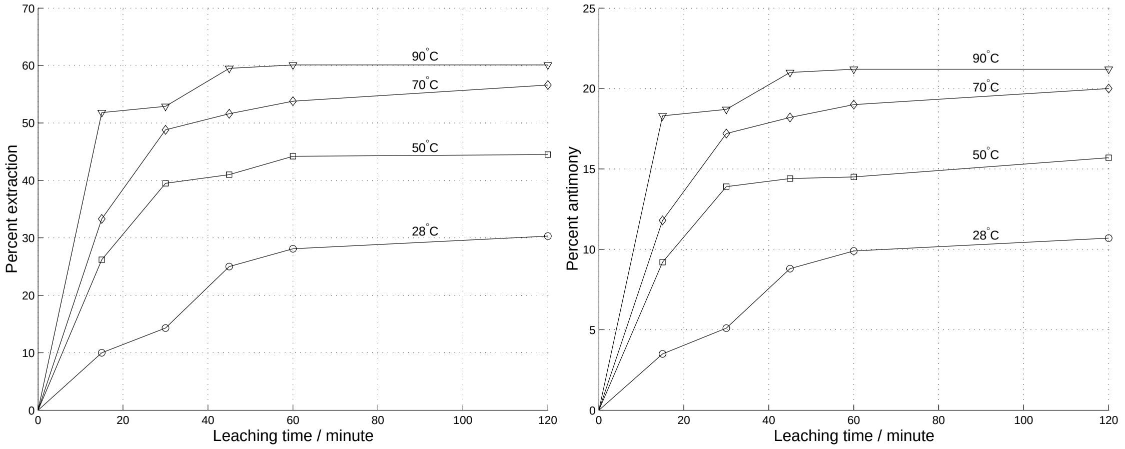
3.1 Influence of the leaching temperature and time

3.1.1 Efficiency of the extraction when leaching the ore in 7N HCl 100 ml.

Time	Extraction temperature							
(minute)	28°C		50°C		70°C		90°C	
	% Sb	% Extr.	% Sb	% Extr.	% Sb	% Extr.	% Sb	% Extr.
15	3.5	10.0	9.2	26.2	11.8	33.3	18.3	51.8
30	5.1	14.3	13.9	39.5	17.2	48.8	18.7	52.9
45	8.8	25.0	14.4	41.0	18.2	51.6	21.0	59.5
60	9.9	28.1	14.5	44.2	19.0	53.8	21.2	60.1
120	10.7	30.3	15.7	44.5	20.0	56.6	21.2	60.1

Table 4 *Experiment results at various time and temperature of extraction.*

From Table 4 and Figure 4 we find that the efficiency and the rate of extraction increase when the extraction temperature increases from 28°Cto 50, 70, and 90°Crespectively. Both the efficiency and the rate of extraction become constant at temperature higher than 70°C. This means that the optimal extraction temperature must be within the range of 70–90°C.



(a) (b)

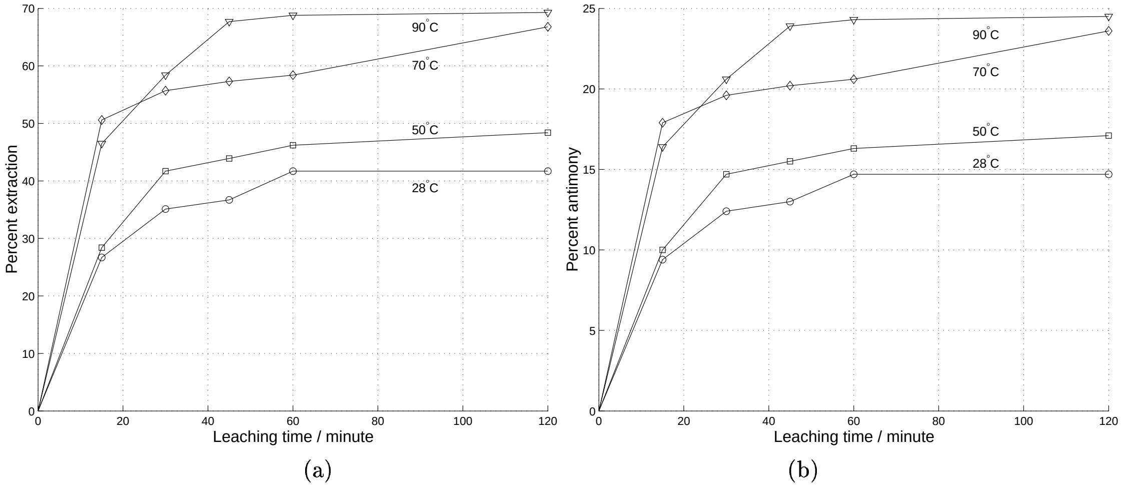
Figure ℵ *Percent extraction (a), and antimony (b), when the leaching solution contains 100 ml 7N HCl and 38.93 g* $\mathrm{FeCl_3 \cdot 6H_2O}$.

3.1.2 Efficiency of the extraction when leaching the ore in 7N HCl 300 ml.

Time	Extraction temperature							
(minute)	28°C		50°C		70°C		90°C	
	% Sb	% Extr.	% Sb	% Extr.	% Sb	% Extr.	% Sb	% Extr.
15	9.4	26.7	10.0	28.4	17.9	50.6	16.4	46.5
30	12.4	35.1	14.7	41.7	19.6	55.7	20.6	58.4
45	13.0	36.7	15.5	43.9	20.2	57.3	23.9	67.7
60	14.7	41.7	16.3	46.2	20.6	58.4	24.3	68.8
120	14.7	41.7	17.1	48.4	23.6	66.8	24.5	69.3

Table 5 *Experiment results at various time and temperature of extraction.*

From Table 5 and Figure 5 it can be seen that the efficiency and the rate of extraction increase as the extraction temperature increases from 28°Cto 50, 70, [and] 90°Crespectively in a manner similar to the case where the ore is leached in 7N HCl 100 ml. But both the efficiency and the rate of extraction is higher than the previous case by approximately 10 percent. This means that the amount of acid used in the extraction by acid leaching has some influence on the efficiency and the rate of extraction.



(a) (b)

Figure ℵ *Percent extraction (a), and antimony (b), when the leaching solution contains 300 ml 7N HCl and 38.93 g* $\mathrm{FeCl_3 \cdot 6H_2O}$.

Influence of the leach solution concentration

The controlled variables are $\mathrm{FeCl_3 \cdot 6H_2O}$ 38.93 g, the concentration of HCl 7N, and the ground antimony ore 14.29 g.

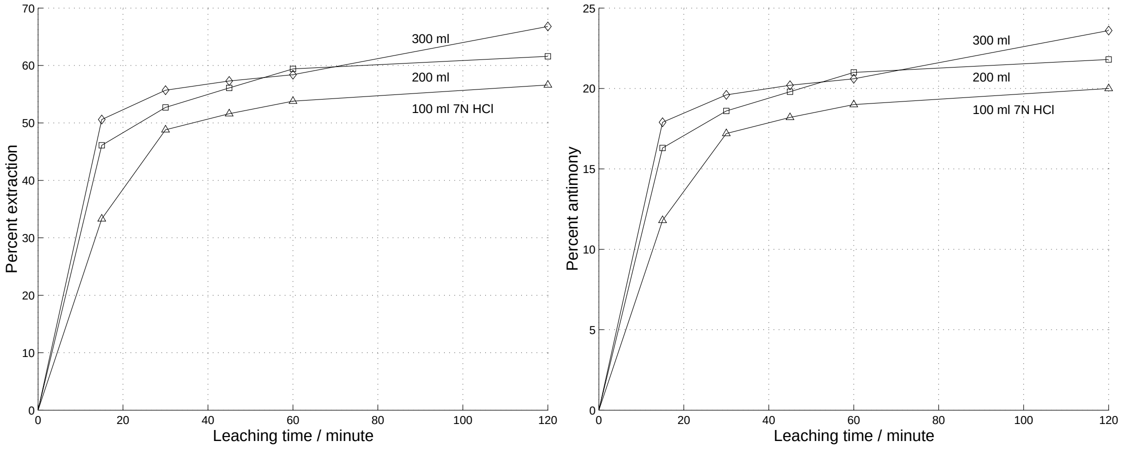
From the data in Table 4 and Table 5 we can draw Figure 6 (28°C), 7 (50°C), 8 (70°C), and 9 (90°C). [Figure 5 to 9, omitted here, compare between 7N HCl 100 and 300 ml] It can be seen that when the amount of acid used is increased from 100 to 300 ml at the extraction temperatures of either 28, 50, 70, or 90°C, the efficiency and the time of extraction invariably increase by approximately 10 percent.

When further experiments are done at the temperature of 70°C, which is a suitable temperature for the extraction in practice, the results obtained are the following. The controlled variables are 7N HCl 200 ml, $FeCl_3 \cdot 6H_2O$ 38.93 g, and the extraction temperature 70°C.

Time (minute)	15	30	45	60	120
% Sb	16.3	18.6	19.8	21.0	21.8
% Extraction	46.1	52.7	56.1	59.4	61.6

Table 6 *Experimental results at 70° C, 200 ml 7N Hcl.*

From Figure 10 one finds that the efficiency and the rate of extraction at 70°Cand the acid concentration 7N increase by approximately 5 percent when the acid amount increases from 100 ml to 300 ml. This means that the efficiency increases as more acid is used.



(a) (b)

Figure ℵ *The effect of various amount of HCl on the extraction efficiency. Percent extraction (a), and percent of antimony (b) in the solution which contains 7N HCl 100, 200, and 300 ml, and 38.93 g* $FeCl_3 \cdot 6H_2O$*, at 70° C.*

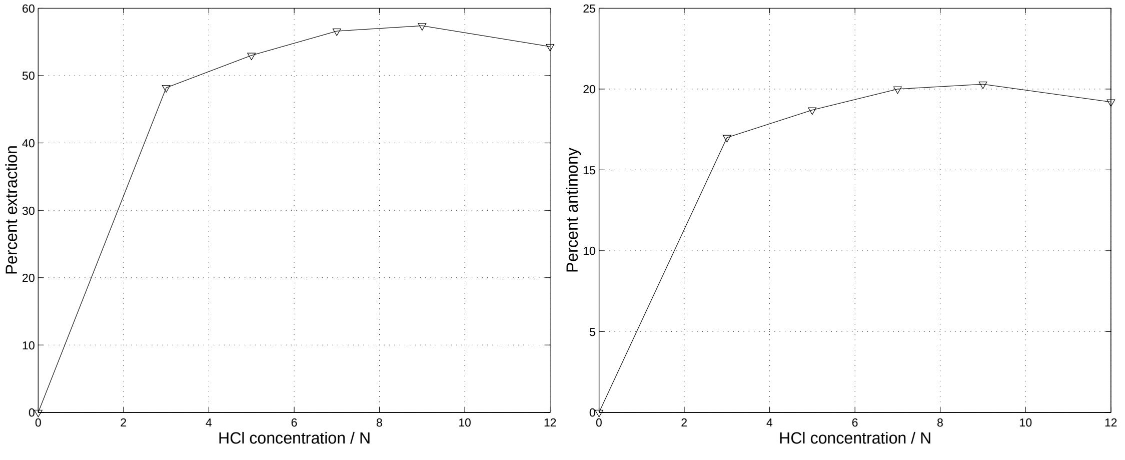
3.3 Influence of the acid concentration used in leaching

The controlled variables are $FeCl_3 \cdot 6H_2O$ 38.93 g, HCl 100 ml, the extraction temperature 70°C, and the time of extraction 2 hours. Vary the concentration of HCl to be 3, 5, 7, 9, and 12N.

Concentration (N)	3	5	7	9	12
% Sb	17.0	18.7	20.0	20.3	19.2
% Extraction	48.2	53.0	56.6	57.4	54.3

Table 7 *Experimental results at various acid concentrations.*

From Table 7 and Figure 11, the efficiency of extraction is highest between the acid concentration 7 and 9N, that is the suitable concentration in practice is 7N.



(a) (b)

Figure 11 *Effect of the acid concentration on the leaching efficiency, (a) the percent extraction, and (b) the percent antimony. The leach solution contains 100 ml HCl and 38.93 g* $FeCl_3 \cdot 6H_2O$at70°C.

When precipitating Sb_2O_3 at the room temperature and at the temperature 90°C, and then take the SEM pictures of the precipitates, the results are shown in Figure 12 and 13 [omitted]. From these it can be seen that when precipitating at a low temperature, the structure of Sb_2O_3 tends to be more cubic than when precipitating at a higher temperature. Therefore if Sb_2O_3 of a cubic structure is desired, the precipitation should be done at a low temperature, while on the other hand if Sb_2O_3 of a rhombic structure is desired, it should be done at a high temperature.

4. Conclusion and suggestions

From the experiments studying the influence of variables in the preparation of Antimonytrioxide from stibnum ores by hydrometallurgical method with the leaching of the ore in an acid solution of ferric chloride it was found that the production of Sb_2O_3 by hydrometallurgical process on the industrial scale has a reasonably high viability when leaching the mineral solution in the hydrochloric acid solution of ferric chloride the acidic concentration of which is between 5–9N HCl at the temperature range of 70–90°C for approximately 1–1.5 hours and with $FeCl_3$ about 10–20% in excess.

Acknowledgements

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§ E.2 Cyberspace

Cyberspace

Kittisak Tiyapan†

Cyberspace is an interesting word that originated from a science-fiction novel by William Gibson. It was adopted by John P. Barlow for the first time to describe the nexus between computers and communication network.

Cyberspace has found its place in our everyday lives and has increased rapidly both in magnitude and in importance. Generally speaking, it conveys the idea of a kind of world created by interconnection between computer systems and telecommunication network. We may think of it as another dimension apart, if not totally separated, from this earth that we live in. It is a kind of virtual reality in its own right.

In this new environment created by communication systems, to mention just a few of which are Internet and CommerceNet, there are features and structures (or virtual structures) that make it a unique world in its own right. It is populated by people (more than 20 million on Internet alone, growing at a rate not less than one million new users per month), it has got addresses, societies, even libraries and educational institutions.

There are businesses going on around the clock sending back and forth not only correspondences but also quotations, invoices and other documents both important and confidential. There are bad guys trying every possible way to do nasty things, and there are also good guys outwitting themselves just trying to catch those that have done bad things or merely just to prevent them from doing so. There are thieves, polices, security officers, etc., as there exist in the real world.

No matter how we view it, this is another dimension. It is not real, in the sense that we use in our four-dimension world, but it is alive. It was created, it is growing fast, it may need a lot of effort for organization (and I assure you that this is by far an easy task), but it is here to stay.

Recommended book

Bruce Sterling. *The Hacker Crackdown. Law and Disorder on the Electronic Frontier.* Penguin Books, 1992.

Interesting terms

Cyberspace. The notional environment within which electronic communication occurs, especially when represented as the inside of a computer system; space perceived as such by an observer but generated by a computer system and having real existence; the space of virtual reality. (cyberpunk + space) (Shorter Oxford Dictionary, 1993)

Cyberpunk. A genre of science fiction that features rebellious computer hackers and is set in a dystopian society integrated by computer networks. (cyber(netics) + punk) (Collins English Dictionary, 3rd ed., 1991)

Cybernetics. The branch of science concerned with control systems in electronic and mechanical devices and the extent to which useful comparisons can be made between man-made and biological systems. (comes from a Greek word which means *steersman*) (Collins English Dictionary, 3rd ed., 1991)

Cyberphobia. Irrational fear of computer.

Cyborg. An integrated man-machine systems. (cybernetics + organism)

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§ E.3 The End of Stars

Field: Astronomy

The End of Stars

Kittisak Tiyapan†

> 'Star light, Star bright,
> I wish I may, I wish I might
> Have a wish I wish tonight.
> I wish I . . .'

This is only one of our many childhood fantasies. After we have grown up, though some of us still say this at night once in a while, our imagination may lead us to wonder further into that infinite depth of the universe.

There have been many theories about how stars are formed and about the evolution of stars. One theory which is widely accepted is that in the beginning there was nothing but a point called *Singularity.* After that a huge outburst of mass and energy or *Big-Bang* occurred. The universe expanded.

After expanding for quite a while, galaxies and stars were formed as a result of non-uniformity in the density of gases. These gases came together because of gravity, got more massive and thus attract more gases. Molecules came close to one another until a point was reached when a fusion process started. The fusion process of a typical star starts from Hydrogen being fused to become Helium. When there is enough Helium in the core, Helium atoms are fused and Carbon atoms formed. The similar processes occur with Carbon, Oxygen, Neon, Magnesium, Silicon and Iron respectively. Iron-fusion does not produce energy so it never occurs. At this stage the star is said to have passed its prime. It produces less and less energy. Temperature reduces. Outer-surface that has previously been bloated out far-and-beyond by the nuclear fusion withdraws itself because of the star's gravity.

There are many ways for a star to end its life. If a star is small, then it dies. But if it is huge to start with, its faith is much more tragic. For example, a star with mass 100-times that of the sun will contract gradually until a critical point is reached. At this point the outer surface of the core bounces back a little while its neighbour outer layers keep moving inward. There is a shock-wave. An explosion occurs where outer-layer which comprises mainly of gases is expelled outward. This is a Super-nova. After this the remaining core contract so rapidly that it never stop. At last it becomes only a mere point in space without any dimension called *Singularity.* A black hole is formed.

Although there are controversies about what implications resulted from the existence of black-holes, scientists believe that they exists. Some claims that black-hole is an inherent time-machine. Some says there exist worm-holes connecting two black-holes located at different corners of the universe acting much like a short cut across space.

Nature is much greater than human beings. We are only a speck of dust in this universe. What will be in your mind when you look up at the stars tonight to make your wish ?

Recommended books

Stuart L. Shapiro and Saul A. Teukolsky. *Black holes, white dwarfs, and Neutron stars – The physics of compacts objects.*

Joseph Silk. *The Big Bang.*

The evolutionary process of stars

- Hydrogen is burnt and Helium is formed
- Helium forms the core because it is heavier
- Star expands as a result of the fusion process
- Amount of Hydrogen lessen
- Inner-core contracts. Temperature and pressure increase
- Fusion of Helium starts
- Carbon resulted from Helium fusion forms inner core
- Fusion of heavier atoms: O, Ne, Mg, Si, Fe respectively
- Iron does not fuse
- Stars with: mass $<$ four-times that of our sun, become white-dwarfs. (Nova) Mass between four- and eight-times of the sun, blow itself up completely, core and all; mass between eight-and fifty-times of the sun, blow outer part, core forms neutron star. (Super-nova) Mass $>$ fifty-times that of the sun, blow outer part, core becomes black-hole.

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Differences between stars and planets

- Stars shrink in size when its temperature is reduced, planets do not.
- There is fusion process in stars but not in planets.

Terms

White-dwarf. Dead star with very density. Sustained by atom-decomposition force.

Neutron-star. Dead star which comprise of pure neutron. Sustained by neutron-decomposition force.

Pulsar. Strong periodic source of magnetic-field. Produced by a Neutron-star.

Quasar. Strong X-ray source. Produced by a Black-hole.

§ E.4 Self-tuning Extremum Control

Design Exercise Report on

Self-Tuning Extremum Control

by Kittisak Tiyapan
under the supervision of Dr M B Zarrop

This report is a part of the MSc Course in Control and Information Technology, 1994/95, Control System Centre,University of Manchester Institute of Science and Technology, Manchester M60 1QD. **abstract**

Self-tuning extremum control is useful when dealing with systems with performance index measured in a noisy environment. This design-exercise studies the approach using the model-based hill-climbing technique. The recursive least square (RLS) algorithm is used. The model used is a quadratic one. The idea is then used in object-location.

1. Introduction

1.1. Important terms

The term *extremum control* (Wellstead and Zarrop, 1991) is a synonym for optimization. It means making a process operate at the optimal point at all times. The optimal point is normally obtained by extremizing a performance index.

Hill-climbing is a very useful technique. We can imagine a performance function as being a hill. Figure 10.1 shows an example of a performance hill.

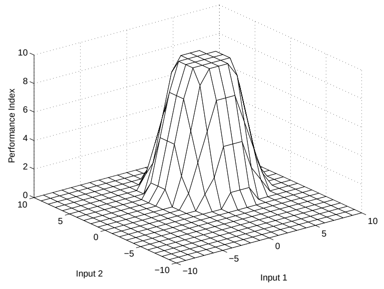


Figure 10.1 Example of a relationship between a performance index and two inputs.

When we are at a specific point (control value) we can calculate the performance index at that point. The performance index we obtain represents the height of the hill there. If we compare this performance index with those of the points in our close neighbourhood we will know which direction to move to in order to increase our performance index. The process is carried out recursively until we reach the top of the hill where no neighbouring point leads to an improvement in performance.

Self-tuning control (Wellstead and Zarrop, 1991) is closely related to adaptive control. Here the input and output of the system are measured and used to adjust model parameters.

When there exists some reference model describing the performance, the output from the reference model, together with measured input/output data can be used to monitor system parameters. This is called a *model-based* algorithm (Wellstead and Zarrop, 1991).

1.2. Objectives

- Investigate various hill-climbing algorithms.
- Apply algorithms to multi-input case.
- Apply algorithms to object location problem.

1.3Overview of tasks

Performance optimization can be done using hill-climbing technique. The environment is usually noisy where the noise is either sensor noise or represents the roughness of the hill. In the model-based approach used here a quadratic model is chosen. It is possible to reduce some parameters in the model (Wellstead and Zarrop, 1991).

With the recursive least squares algorithm it is possible to extend the approach to multi-input case (Zarrop and Rommens, 1993) and hence to an object-location problem.

The system noise is assumed to be a gaussian random variable with zero mean and adjustable variance. To prevent the estimator from *sleeping* a dither signal is added which, in this case, is a zero-mean uncorrelated process, uniformly distributed on a chosen interval.

2. Simulations

2.1. Quadratic model

2.1.1. Introduction

In the neighbourhood of extremum a quadratic model is a good representation of the hill top. If the hill is smooth at the summit, this model represents the first few terms of a Taylor's series for the performance function. The model chosen is

$$\text{M1:} \qquad y = au^2 + bu + c \tag{1}$$

where y is the performance index to be maximized and u is the adjustable factor (control input). If $a < 0$, then (1) represents a hill (not a valley) with a summit at

$$u = -\frac{b}{2a} \tag{2}$$

The true relationship between u and y is only approximated by (1). The true hill is usually mapped out empirically but, for simulation purposes, we may assume that the *data generator* is given by

$$y = g(u) + \text{noise} \tag{3}$$

where the noise represents either sensor noise or reflects the *roughness* of the hill. In simulations, (3) is implemented as a subroutine and the noise is generated as a gaussian random variable with zero mean and adjustable variance.

An iterative algorithm for finding the hill summit is as follows. (Here t denotes the iteration count and the input $u(t-1)$ gives rise to the next output y(t).)

2.1.2. Quadratic model algorithm (Model M1)

Initial Conditions Set $t = 0$, $\hat{a}(0)$, $\hat{b}(0)$, $\hat{c}(0)$, $\mathbf{P}_{(3\times3)}(0)$

Step 1 Calculate $u(t) = -\frac{\hat{b}(t)}{2\hat{a}(t)} + v(t)$ **(4)**

Step 2 $t = t + 1$ **(5)**

$y(t) = g(u(t-1)) + e(t)$ **(6)**

Step 3 $\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I}_3 - \frac{\mathbf{x}(t)\mathbf{x}^T(t)\mathbf{P}(t-1)}{\lambda+\mathbf{x}^T(t)\mathbf{P}(t-1)\mathbf{x}(t)}\right\}$ **(7)**

$\hat{\theta}(t) = \hat{\theta}(t-1) + \mathbf{P}(t)\mathbf{x}(t)\left\{y(t) - \mathbf{x}^T(t)\hat{\theta}(t-1)\right\}$ **(8)**

Step 4 Goto Step 1

NB $\hat{\theta}(t) = \left[\hat{a}(t), \hat{b}(t), \hat{c}(t)\right]^T$

$\mathbf{x}(t) = \left[u^2(t-1), u(t-1), 1\right]^T$

Remark In (4) the current parameter estimates are inserted in (2) to give an estimate of the optimal input.The additional *dither* signal $v(t)$ is used to excite the system, so that the estimator does not *go to sleep*. The dither signal is chosen as a zero mean uncorrelated process, uniformly distributed on a chosen interval.

2.1.3. Quadratic model simulation

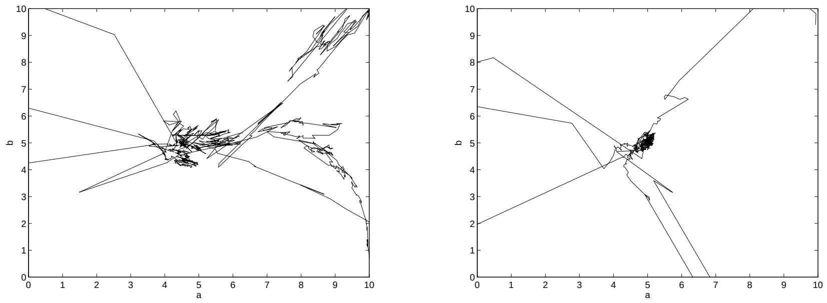
Try the algorithm using various conditions and with different types of hill. Estimating hills using a quadratic model.

Basis: $\lambda = 0.98$, $\sigma_v^2 = 1.0$, $\sigma_e^2 = 0.5$ (λ = forgetting factor, σ_v^2 = dither signal's variance, σ_e^2 = noise's variance)

2.1.4. Results

The result does not depend on initial position of estimated hill top.

When λ *= 0.10, 0.50, 0.70, 0.90, 0.95, 0.98, 0.99, 1.0, 3.0 were used.* λ between 0.95 and 1.0 gives good result. Smaller λ produces much uncertainty. When $\lambda > 1$ is used the result does not converge to the true value. *When* σ_v^2 *= 0.1, 0.3, 0.5, 0.7, 0.9, 1, 3, 5, 7, 9 were used.* Time of convergence is generally approximately 50 time steps. When σ_v^2 is small, the parameters' values estimated will scatter around the true value. This can be easier illustrated using a plot between two parameters. Figure 2 shows a comparison between two different values of σ_v^2's.



(a) $\sigma_v^2 = 0.01$ (b) $\sigma_v^2 = 1$

Figure 2 *Plot of* $\hat{b}(t)$ *vs* $\hat{a}(t)$. *(a)* $\sigma_v^2 = 0.01$, *(b)* $\sigma_v^2 = 1$.

When various hill types are used. (Hills used : circular-top, sinusoidal, third order polynomial, fourth order polynomial, sinc function, symmetric piece-wise functions, asymmetric piece-wise function, pulse shaped function.) Convergence to the local extremum that is closest to starting point. For badly conditioned hill, for example a pulse shaped function, convergence does not always occur within 300 time steps. Bias exists with all non-symmetric hills.

[In] Figure 3 When both system and model are quadratic estimation is most perfect. Here we have the true hill (top left). We start off with a quadratic estimating valley and end up with exactly the same hill (top right). Control input is plotted against number of steps (bottom left). Estimated parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$ are plotted to show the convergence process during estimation (bottom right).

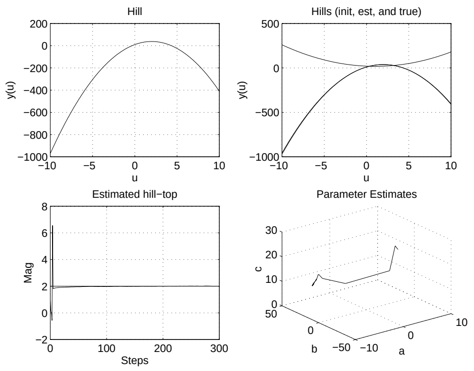


Figure 3. Model M1 with quadratic system and model.

[In] Figure 4 This is the case where the true hill is sinusoidal. Here we have the true hill (top left). We start off with a quadratic estimating valley and end up with another valley (top right). Control input is plotted against number of steps (bottom left). Notice that in this case the hill top estimated turns out to be a local minimum. Estimated parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$ are plotted to show the convergence process during estimation (bottom right).

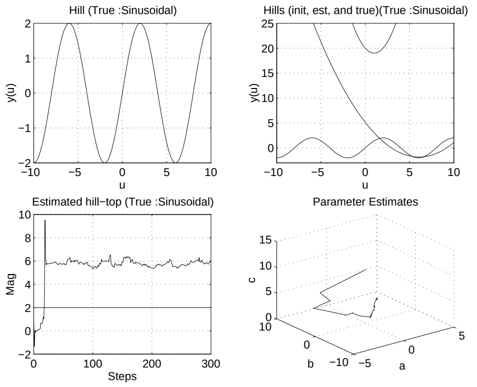


Figure 4. Model M1 with sinusoidal-shaped hill.

[In] Figure 5 This is the case where the true hill is a third order polynomial. Here we can see a plot of the true hill (top left). We start off with a quadratic estimating valley and end up with a hill whose top is close to the local maximum at -1 (top right). Notice the bias that exists in the hill top estimated. Control input is plotted against number of steps (bottom left). Estimated parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$ are plotted to show the convergence process during estimation (bottom right).

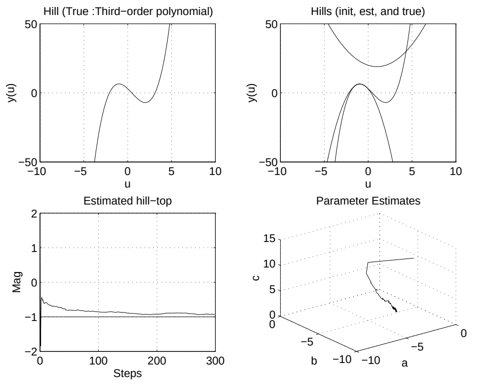
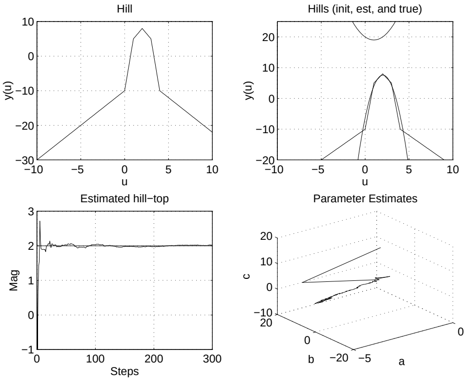
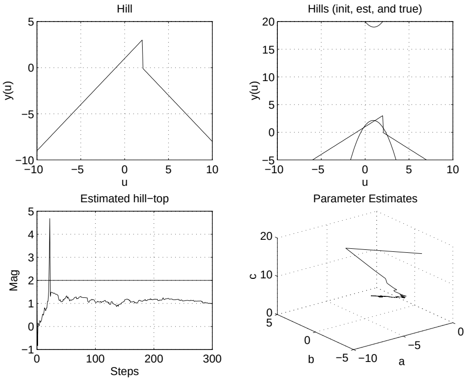


Figure 5. Model M1 with a third order polynomial hill.



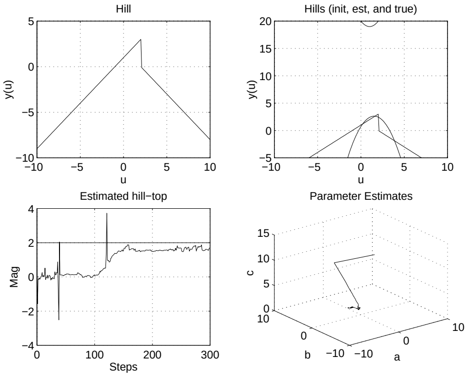
[In] Figure 6 This is the case where the true hill is a symmetric piecewise function of control input. Here we can see a plot of the true hill (top left). We start off with a quadratic estimating valley and end up with a hill whose top represents exactly the position of the true hill top (top right). Thus for a symmetric hill the algorithm does very well. Control input is plotted against number of steps (bottom left). Estimated parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$ are plotted to show the convergence process during estimation (bottom right).

Figure 6. Model M1 with a symmetric piecewise function as a hill.



[In] Figure 7 This is the case where the true hill is an asymmetric piecewise function of control input. Here we can see a plot of the true hill (top left). We start off with a quadratic estimating valley and end up with a hill whose top has considerable bias compared with the true one (top right). Thus for an asymmetric hill the algorithm does produce biased estimate. Control input is plotted against number of steps (bottom left). Estimated parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$ are plotted to show the convergence process during estimation (bottom right).

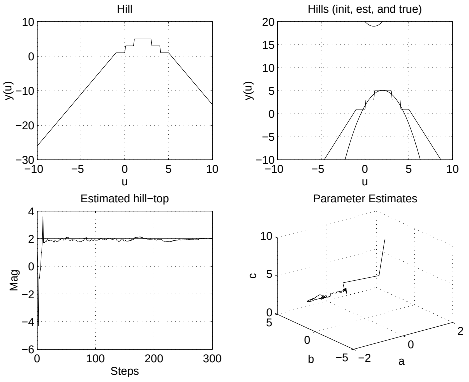
Figure 7. Model M1 with an asymmetric piecewise function as a hill.



[In] Figure 8 We can see a plot of the true hill (top left). We start off with a quadratic estimating valley and end up with a hill whose top has considerable bias compared with the true one (top right). Control input is plotted against number of steps (bottom left). Notice that the bias is now reduced. Thus smaller dither signal produce less bias. Estimated parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$ are plotted to show the convergence process during estimation (bottom right).

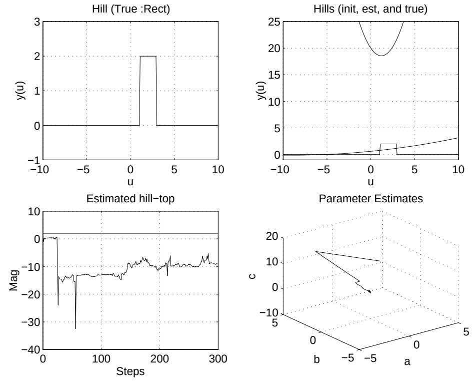
Figure 8. Model M1 with an asymmetric piecewise function and small dither signal variance.

Here (Figure 8) the hill is the same one as Figure 7. Smaller dither signal is used ($\sigma_v^2 = 0.05$ compared to $\sigma_v^2 = 0.5$ in Figure 7).



[In] Figure 9 The hill now has got flat parts and its top is also flat. The hill itself is symmetric. The result has got some bias but converge to the position of the hill top. We can see a plot of the true hill (top left). We start off with a quadratic estimating valley and end up with a hill (top right). Control input is plotted against number of steps (bottom left).

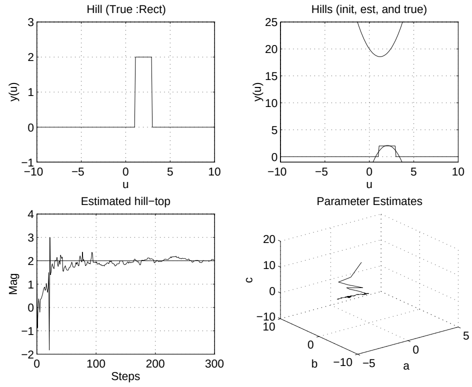
Figure 9. Model M1 with a hill that has flat top.



[In] Figure 10 We can see a plot of the true hill (top left). We start off with a quadratic estimating valley and end up with a hill (top right). Control input is plotted against number of steps (bottom left). Estimated parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$ are plotted to show the convergence process during estimation (bottom right).

Figure 10. Model M1 with pulse-shaped hill.

The hill in Figure 10 gives no directional information outside its range. The hill itself is symmetric. Sometime the result does converge but here we have shown the case where it does not converge to the true position after 300 time steps.



[In] Figure 11 This is the case where we can get the correct estimation.

Figure 11. Model M1 with pulse-shaped hill (another one).

2.1.4. Discussion

We have investigated the influence of dither signal variance on estimation of the hill top. From (Figure 2) we can see that if the dither signal is small it will take longer before the estimates can reach the final value. Hence the phase plane shows a larger hazy patch around the true value of states.

The quality of estimation depends on many factors. If the true hill is badly conditioned, as an example a non-symmetric hill, there may be some bias or some difficulties in convergence. Also if the hill rises from a flat plane, the estimating procedure might get lost when not on the hill. Here is where a random search or some other procedure must be introduced to find the hill again.

2.1.5. Summary

A quadratic model is a suitable model for estimating a hill top because it is a good representation of the summit in the close neighbourhood of extremum. If we choose our dither signal carefully we will be able to get the accuracy required provided that the environment is not too noisy and the hill is reasonably well conditioned.

2.2. Reduced-parameter model

2.2.1. Introduction

The number of estimation parameters can be reduced from three to one. Note that (4) does not require $\hat{c}(t)$ and this parameter can be removed from (1) by differencing the data as follows. Let

$$\Delta y(t) = y(t) - y(t-1) \tag{9}$$

then

$$\text{M2:} \qquad \Delta y(t) = a\Delta u^2(t-1) + b\Delta u(t-1) \tag{10}$$

Secondly we may fix $a = a_f < 0$ and (10) may be written as

$$y'(t) = \mathbf{x}^T\theta \tag{11}$$

where

$$y'(t) = \Delta y(t) - a_f \Delta u^2(t-1) \tag{12}$$

$$\mathbf{x}(t) = \Delta u(t-1) \tag{13}$$

$$\theta = b \tag{14}$$

Then, based on the model

$$\text{M3:} \quad \Delta y(t) = a_f \Delta u^2(t-1) + b \Delta u(t-1) \tag{15}$$

The basic recursive algorithm is as before with the modifications

Initially choose $\hat{b}(0)$ and scalar $P(0)$
(4) is replaced by (16)
RLS is based on the model (11) - (14)

Note that, because a_f may not be the true value of the curvature parameter, the success of the algorithm depends on $\hat{b}(t)$ converging to a value b^* such that $-\frac{b^*}{2a_f}$ is still the correct input to achieve the summit.

2.2.2. Reduced-parameter model algorithm

Initial Conditions Set $[t = 0]$; $a_f = -7$, $\hat{b}(0)$, $\hat{b}(1)$, $P(1)$

Step 1 Calculate $u(t) = -\frac{\hat{b}(t)}{2a_f} + v(t)$ **(16)**

Step 2 $t = t + 1$ **(17)**

$y'(t) = bx(t) + e(t)$ **(18)**

Step 3 $P(t) = \frac{1}{\lambda} P(t-1) \left\{ 1 - \frac{x^2(t)P(t-1)}{\lambda + x^2(t)P(t-1)} \right\}$ **(19)**

$\hat{b}(t) = \hat{b}(t-1) + P(t)x(t) \left\{ y'(t) - \hat{b}(t-1)x(t) \right\}$ **(20)**

Step 4 Goto Step 1

NB $x(t) = \Delta u(t-1) = u(t-1) - u(t-2)$ **(21)**

2.2.3. Reduced-parameter model simulation

Investigate the nature of reduced-parameter algorithm in estimating hill-top position. By discarding c and holding a constant $(= a_f)$ only the b parameter is estimated (Wellstead, 1991).

Basis: $\lambda = 0.98$, $\sigma_e^2 = 1.0$, $\sigma_v^2 = 0.5$, $a_f = -7$

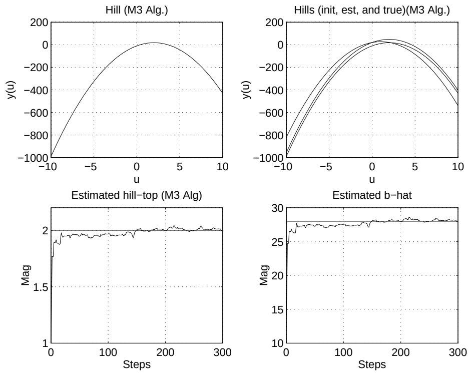
2.2.4. Results

The result does not depend on initial position of estimated hill top.

When $\lambda =$ *0.10, 0.50, 0.70, 0.90, 0.95, 0.98, 0.99, 1.0, 3.0 were used.* λ between 0.95 and 1 gives good result. Smaller λ produces much uncertainty.

When $\sigma_v^2 =$ *0.1, 0.3, 0.5, 0.7, 0.9, 1, 3, 5, 7, 9 were used.* Larger variance gives better estimation, ie. less uncertainty and faster estimation time.

When $\sigma_e^2 =$ *0.1, 0.3, 0.5, 0.7, 0.9, 3, 5, 7, 9 were used.* When σ_e^2 increases, uncertainty in estimation increases.



[In] Figure 12 The hill used here is quadratic. We estimate only one parameter (ie. b). We can see a plot of the true hill (top left). We start estimating our hill arbitrarily and end up with a correct hill top position (top right). Notice that the estimated hill and the true hill are not exactly the same but their hill top positions are the same. Control input is plotted against number of steps (bottom left). Estimated parameters $\hat{a}$, $\hat{b}$, and $\hat{c}$ are plotted to show the movement during estimation procedure (bottom right).

Figure 12. Model M3 with a quadratic hill.

2.2.5. Discussion

The reduced-parameter quadratic model seems to work better than the model with all three parameters being estimated. This must be because we assume that our a parameter is a fixed value, ie. we assume that we have more knowledge about the system.

The effects of various variables, for example σ_v^2 and σ_e^2 on the estimation is the same as those of simulation using the model M1 described in the last chapter. At this stage we have not done further study in using the model to different shapes of hill. It would be an interesting thing to do.

2.2.6. Summary

We investigate briefly the quadratic model with reduced parameters. We have verified that it is possible to discard the c parameter and to fix the a parameter in our model.

2.3. Multi-input model

2.3.1. Introduction

In the case when the control input is a vector (ie. we have a number of adjustable factors that influence performance) (1) can be generalised to

$$y = \mathbf{u}^T A \mathbf{u} + \mathbf{b}^T \mathbf{u} + c \tag{22}$$

where $\mathbf{u}$ is the control vector (of dimension m) and A is a symmetric matrix $(m \times m)$. If we estimate all the parameters in this model, we will have $\frac{1}{2}(m+1)(m+2)$ of them (for $m = 1$ this gives 3). We assume the model that is equivalent to (15), ie. fix A and use data differencing:

$$\text{M3}' : \qquad \Delta y(t) = -\sum_{i=1}^{m} \Delta u_i^2(t-1) + \sum_{i=1}^{m} b_i \Delta u_i(t-1) \tag{23}$$

This corresponds to choosing the fixed A as

$$\mathbf{A}_f = -\mathbf{I}_m \tag{24}$$

(23) can then be written in the form (11), where

$$y'(t) = \Delta y(t) + \sum_{i=1}^{m} \Delta u_i^2(t-1) \tag{25}$$

$$\mathbf{x}(t) = [\Delta u_1(t-1), \cdots, \Delta u_m(t-1)]^T \tag{26}$$

$$\theta = \mathbf{b} \tag{27}$$

In this model there are only m parameters to be estimated.

The control synthesis (4) (Step 1 of the algorithm) follows from (22) by calculating $\mathbf{u}$ to maximize the quadratic, ie.

$$\mathbf{u} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b} \tag{28}$$

Introducing the assumption (24) and adding dither we get

$$\mathbf{u} = \frac{1}{2}\mathbf{b} + \text{dither} \tag{29}$$

i.e. $u_i(t) = \frac{1}{2}\hat{b}_i(t) + v_i(t), \qquad i = 1, \cdots, m$ (30) Note that there is a different dither component on each control component.

2.3.2. Multi-input model algorithm (Model $M3'$)

The model used is one with two inputs. Here we do not consider the c parameter and do not take in to account the a parameter.

Initial Conditions Set $[t=0]$; $\mathbf{A}_f = -\mathbf{I}_2$, $\hat{b}_1(0)$, $\hat{b}_1(1)$, $\hat{b}_2(0)$, $\hat{b}_2(1)$, $\mathbf{P}_{2\times 2}(1)$

Step 1 Calculate $u_1(t) = \frac{\hat{b}_1(t)}{2} + v_1(t)$ **(31)**

$u_2(t) = \frac{\hat{b}_2(t)}{2} + v_2(t)$ **(32)**

Step 2 $t = t + 1$ **(33)**

$y(t) = g(\mathbf{x}(t)) + e(t)$ **(34)**

Step 3 $\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I} - \frac{\mathbf{x}(t)\mathbf{x}^T(t)\mathbf{P}(t-1)}{\lambda + \mathbf{x}^T(t)P(t-1)\mathbf{x}(t)}\right\}$ **(35)**

$\hat{\theta}(t) = \hat{\theta}(t-1) + \mathbf{P}(t)\mathbf{x}(t)\left\{y(t) - \mathbf{x}^T(t)\hat{\theta}(t-1)\right\}$ **(36)**

Step 4 Goto Step 1

NB $\mathbf{x}(t) = [\Delta u_1(t-1), \Delta u_2(t-1)]^T$

$\hat{\theta} = \left[\hat{b}_1(t), \hat{b}_2(t)\right]^T$

The next algorithm that we will try is the one which estimates nearly all the parameters (ie. a's, b's, and c's). Here we only fix some of the parameters in the A matrix. We are going to use this model for the object-location in the next chapter.

Initial Conditions Set $[t=0]$; $\hat{a}_1(0)$, $\hat{a}_2(0)$, $\hat{b}_1(0)$, $\hat{b}_2(0)$, $\hat{c}_2(0)$, $\hat{c}_2(0)$, $\mathbf{P}_{6\times 6}(0)$

Step 1 Calculate $p_1(t) = \frac{\hat{b}_1(t)}{2\hat{a}_1} + v_1(t)$ **(37)**

$q_2(t) = \frac{\hat{b}_2(t)}{2\hat{a}_2} + v_2(t)$ **(38)**

Step 2 $t = t+1$ **(39)**

$y(t) = g(\mathbf{x}(t)) + e(t)$ **(40)**

Step 3 $\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I}_6 - \frac{\mathbf{x}(t)\mathbf{x}^T(t)\mathbf{P}(t-1)}{\lambda+\mathbf{x}^T(t)P(t-1)\mathbf{x}(t)}\right\}$ **(41)**

$\hat{\theta}(t) = \hat{\theta}(t-1) + \mathbf{P}(t)\mathbf{x}(t)\left\{y(t) - \mathbf{x}^T(t)\hat{\theta}(t-1)\right\}$ **(42)**

Step 4 Goto Step 1

NB $\mathbf{x}(t) = \left[p^2(t-1), q^2(t-1), p(t-1), q(t-1), 1, 1\right]^T$

$\hat{\theta}(t) = \left[\hat{a}_1(t), \hat{a}_2(t), \hat{b}_1(t), \hat{b}_2(t), \hat{c}_1(t), \hat{c}_2(t)\right]^T$

2.3.3. Multi-input model simulation

Use a quadratic hill with two inputs. Study the effects of values of λ, σ_v^2, σ_e^2 and initial estimated hill top.

Basis: $\sigma_e^2 = 1$, $\sigma_{v1}^2 = 1$, $\sigma_{v2}^2 = 0.5$, $\lambda = 0.98$, A is a matrix which all element is one

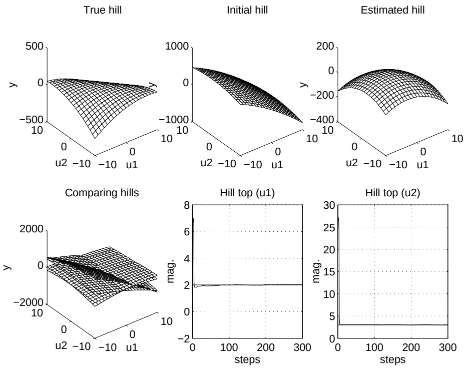
2.3.4. Results

When λ = 0.70, 0.90, 0.95, 0.98, 1.0 were used. $\lambda = 0.98$ and 1 give good result. With smaller value of lambda uncertainty in estimated hill top increases.

When σ_v^2 = 0.1, 3, 7 were used. Greater values of dither variance give more accurate results.

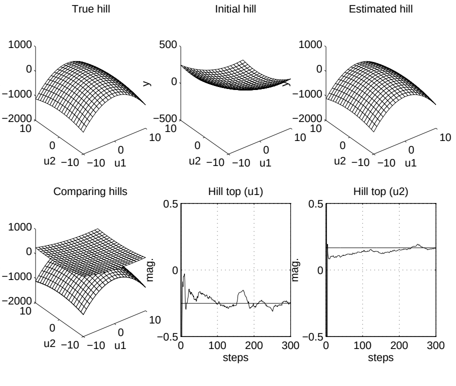
When σ_e^2 = 0.1, 3, 5, 10 were used. Smaller values of σ_e^2 give more accurate results.

When various positions of the hill are used. Accuracy of convergence does not depend on position of the true hill.



[In] Figure 13 The hill used here is quadratic. We estimate only one parameter for each input (ie. b's). We can see a plot of the true hill (top left), the initial hill used for estimating procedure (middle top), and the final estimated hill (top right). We start estimating our hill arbitarily and end up with a correct hill top position. The three hills are then super-imposed (bottom left). The last two plots are the estimated values of hill tops for both the control inputs (bottom middle and right).

Figure 13. Model $M3'$ with two inputs.



[In] Figure 14 The hill used here is quadratic. We estimate all three parameters (ie. a, b, and c). We can see a plot of the true hill (top left), the initial hill used for estimating procedure (middle top), and the final estimated hill (top right). We choose to start with a valley and end up with a hill that has got correct hill top position. The three hills are then super-imposed (bottom left). The last two plots are the estimated values of hill tops for both the control inputs (bottom middle and right). Now we are ready to tackle the object-location problem using this model.

Figure 14. Model $M3'$ with two inputs.

2.3.5. Discussion

Our algorithm does very well when using with a two-input system. Again, this may be because we chose to use the reduced-parameter model. It would be interesting to investigate the same problem using full-parameter model or some other models or methods.

2.3.6. Summary

Often a performance function depends on more than one control value. We have shown that it is possible to extend our RLS algorithm to a multi-input quadratic model (in this case, two-input) with fixed curvature matrix.

2.4. Object-location model

2.4.1. Introduction

We may be interested in locating an object within an area (see figure 15). We consider that the object is specified by the (x, y) locations of a number of pixels within the image itself. We think of the location process as a camera moving across the image and until the number of pixels seen by the camera are maximized in number. We model this as a square mask moving in the $x - y$ plane (without rotation, for simplicity) until the number of pixels within the mask is at maximum.

The model which will be used in the following algorithm has been introduced earlier in the last chapter.

2.4.2. Object-location model algorithm

```
Initial Conditions Set $t = 0$; $\hat{a}_1(0)$, $\hat{a}_2(0)$, $\hat{b}_1(0)$, $\hat{b}_2(0)$, $\hat{c}_1(0)$, $\hat{c}_2(0)$, $\mathbf{P}_{6\times 6}(0)$
Step 1  Calculate $p(t) = -\frac{\hat{b}_1}{2\hat{a}_1} + v(t)$                                (43)
          $q(t) = -\frac{\hat{b}_2}{2\hat{a}_2} + v(t)$                                        (44)
Step 2  (p,q) is mid-point of the mask
          $y$ = overlapped pixels
Step 3  if $y > 1$ then
            $t = t + 1$                                                                        (45)
            $\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I} - \frac{\mathbf{x}(t)\mathbf{x}^T(t)\mathbf{P}(t-1)}{\lambda + \mathbf{x}^T(t)\mathbf{P}(t-1)\mathbf{x}(t)}\right\}$   (46)
            $\hat{\theta}(t) = \hat{\theta}(t-1) + \mathbf{P}(t)\mathbf{x}(t)\left\{y(t) - \mathbf{x}^T(t)\hat{\theta}(t-1)\right\}$   (47)
        else random search for $p$ and $q$
            $\hat{b}_1 = -2a_1 p$                                                              (48)
            $\hat{b}_2 = -2a_2 q$                                                              (49)
Step 4  Goto Step 1
    NB    $\mathbf{x}(t) = \left[p^2(t-1), q^2(t-1), p(t-1), q(t-1), 1, 1\right]^T$
          $\hat{\theta}(t) = \left[\hat{a}_1(t), \hat{a}_2(t), \hat{b}_1(t), \hat{b}_2(t), \hat{c}_1(t), \hat{c}_2(t)\right]^T$
```

2.4.3. Object-location model simulation

Locating objects of various shapes and sizes at unknown locations in a predefined area. Try to match our image with the real object. Both the object and the image are two-dimensional. A square shaped image with nine-pixel size is used. The overlapped area measured is subject to white noise.

Basis: $\lambda = 0.98$, $\sigma_v^2 = 0.1$, $\sigma_e^2 = 0.1$, area = 431 pixels, object size = 9 pixels, image size = 9 pixels, N = 1000, square shaped object.

2.4.4. Results

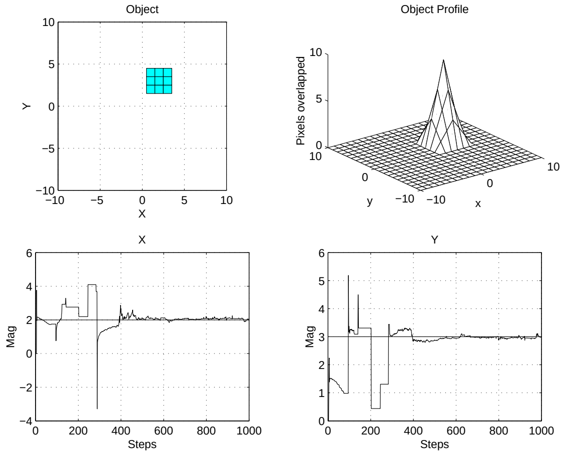
When $\lambda =$ *0.10, 0.30, 0.50, 0.70, 0.90, 0.95, 0.98, 0.99, 1.0, and 2.0 were used.* $\lambda = 0.98$ and 0.99 give satisfactory convergence. With $\lambda = 0.99$ there was some bias present.

When $\sigma_v^2 =$ *0.01, 0.05, 0.1, 0.5, 1, and 5 were used.* $\sigma_v^2 = 0.01$, 0.05, 0.1, 0.5 give satisfactory convergence with some bias.

When $\sigma_e^2 =$ *0.01, 0.05, 0.1, 0.5, 1, and 5 were used.* $\sigma_e^2 = 0.01$, 0.05, 0.1 give satisfactory convergence. Sometime results do not converge after 1000 time steps. But normally convergence occurs after 200 steps.

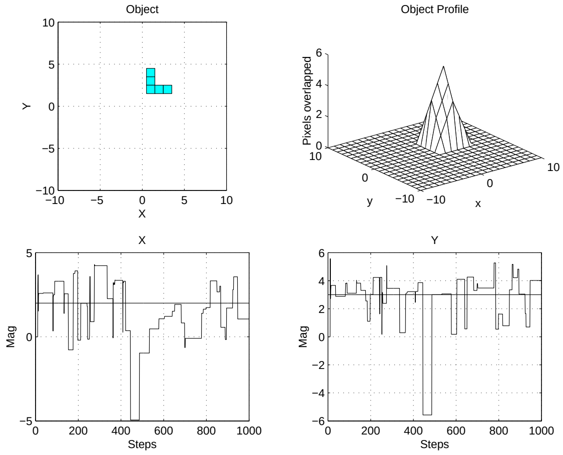
When various object shapes are used. (square of various sizes, rectangulars of various sizes,letter-L shape, letter-O shape, letter-T shape, Plus-sign) When locating objects with a mask smaller in size, estimated position will drift but bounded inside the true object.

There is bias when using asymmetrically-shaped objects.



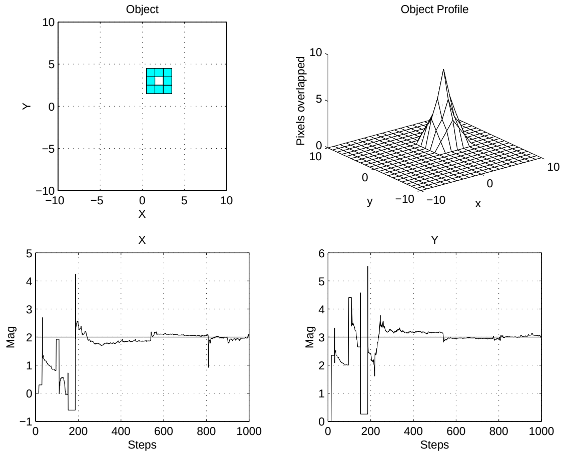
In Figure 15 we have a picture of the object (top left), the hill resulting from counting the number of overlapped pixels (top right), and the two coordinates of mid-point of estimated image (bottom).

Figure 15. Object-location when object is a square of the same size as the image.



In Figure 16 we have a picture of the object (top left), the hill resulting from counting the number of overlapped pixels (top right), and the two coordinates of mid-point of estimated image (bottom).

Figure 16. Object-location when object is shaped like an L letter.



In Figure 17 we have a picture of the object (top left), the hill resulting from counting the number of overlapped pixels (top right), and the two coordinates of mid-point of estimated image (bottom).

Figure 17. Object-location when object is shaped like an O letter.

2.4.5. Discussion

Because we use a random-search method in finding our way when thrown off the hill, the time of convergence does not depend only on the RLS algorithm. There are times when the result does not converge within 1000 time steps. And there is a possibility that one might be thrown off the hill after getting on it. One possible solution to this problem could be to adopt the procedure of *stepping back* when thrown off the hill.

The method of counting pixels can be time-consuming especially when our area of interest gets much larger or when the resolution increases. In real practice this problem may be overcome if we can use an analog measurement of the overlapped area and convert it to digital signal later. For example, consider measuring the intensity of light that passes through two masks.

2.4.6. Summary

Possibility of using the self-tuning extremum control in object-location was investigated. We have shown that the possibility is there. In order to be able to apply the idea to real problems there has to be a further and extensive study on the topic.

3. Discussion

For the Object Location Problem because a random search is used the time of convergence varies and the result may not converge after 1000 time steps.

Quality of convergence depends very much on how well conditioned the hill is. For a symmetric hill the result has no bias.

The purpose of dither signal is to give the essential excitation to the system. Theoretically a smaller dither signal is better. Big dither signals may cause some bias problem when a hill is asymmetric.

The value of the forgetting factor λ must be ≤ 1. $\lambda > 1$ means that the past data is more important than the one now, which is unrealistic. For good results λ should be in the range $0.95 \leq \lambda \leq 1.0$.

4. Conclusion

Performance optimization (Wellstead and Zarrop, 1991) is one of the common objectives. Here we have used a model-based approach with a quadratic model. The result of simulation has shown that it is possible to climb a performance hill in a noisy environment.

Depending on how much knowledge of the hill we have at the start of our quest, we may be able to fix some of the parameters without any adverse effect on the usefulness of the result obtained (Wellstead and Zarrop, 1991).

We may extend our algorithm to systems with more than one input (Zarrop and Rommens, 1993).

It is possible to apply the algorithm to the Object Location Problem. This could be useful in many areas, for example vision-robot and object tracking.

5. Demonstration Program

The program used in doing the simulation has been written in Matlab code and has been adapted to be suitable for use as a demonstration program. It is called *extremum*.

The program is menu-driven. It includes almost everything done in doing this design exercise.

The author intends to write another program in some high-level language, for example C or C++. These languages are more flexible and thus should allow more ease in tailoring the program. Also, one can be free from the availability of the MATLAB program which could cause a problem sometime.

References

P. E. Wellstead and Zarrop, M.B. *Self-Tuning Systems.* John Wiley & Sons. 1991.

M. B. Zarrop and M. J. J. J. Rommens. Convergence of a multi-input adaptive extremum controller. *IEE Proceedings-D.* **140**, 2, 65–69. March 1993.

§ E.5 The story of Andromedra

The story of Andromeda

Kittisak Tiyapan

> Andromeda casts down her sigh
> and Vega lights my way.
> from a song by John Denver

After arriving in Tokyo from England in October, I have found myself studying an intensive Japanese course at the Tokyo Institute of Technology. The course is interesting as well as well organised. We often have study trips on weekends too, for example we went to Kamakura which used to be a prefecture of Japan before Tokyo became a capital of the country.

On this first of November we had another one of such trips again. This time we went to the Tsukuba Expo Centre. Among other things there was one planetarium. We went inside, and this was the story that we listened to on that day. It was from Greek mythology.

One of the stories from Greek mythology which is being referred to very often in European literature, is that of Andromeda. Anyone who reads English literature will surely have come acrossed the story.

Born a daughter to Cepheus, who was a king of Ethiopians, and Cassiopeia (Cassiepeia, Cassiope), Andromeda was a very charming maiden. Her mother, being so proud of her, boasted that she was even more beautiful than the sea nymphs Nereids, daughters of the sea-god Nereus. They made a complaint to Poseidon, the God of the sea, who then flooded the land and sent a sea-monster called Cetus. Cepheus, hoping to lessen the fury of Poseidon, fastened Andromeda to the rock on sea-shore. At that moment Peseus, son of Zeus and Danae, bringing back the head of Medusa whom he had killed, came by. He fell in love with Andromeda and asked Cepheus for her hand. Cepheus said that he would consent to the marriage proposal only if Peseus could get rid of Cetus first, which he did. After killing Cetus he was attacked by an uncle and betrothal of Andromeda, Phineus, whom he then turned into stone by showing his souvenir, the head of Medusa.

Perseus and Andromeda got married and lived together happily. After their death they were turned into constellations in the sky, together with Cepheus, Cassiopeia, and Cetus. Cepheus and Cassiopeia are among the northern constellations; Andromeda, Pegasus, and Cetus are among those of autumn, while Perseus can be seen in the winter sky [2].

In order to give readers some appreciation of the story, we will venture to catch a glimpse at a book by Hugo [1].

> Alas! Will none come to the help of the human soul in this gloom? Is it its destiny forever to await the mind, the liberator, the huge rider of Pegasus and hippogriffs, the aurora-hued combatant who shall descends from the sky with wings, the radiant knight of the future? Shall it always call to its aid the bleaming lance of the ideal in vain? is it condemned to hear the Evil coming terribly through the depth of the abyss, and to see nearer and nearer at hand, under the hideous water, that dragon-head, those jaws reeking with foam, that serpentine waving of claws, distensions, and rings? Must it remain there, with no ray, no hope, abandoned to that horrible approach, vaguely scented by the monster, shuddering, dishevelled, wringing its hands, forever chained to the rock of night, a sombre Andromeda white and naked in the darkness?

Here Hugo used Andromeda as a metaphor for prisoners of the galleys during the tumultuous period that followed the French Revolution, many of whom were unjustly sentenced.

References

[1] Victor Hugo. *Les Misérables.* Wordsworth Editions Limited. 1994. (also another translated version can be found at `gopher://gopher.etext.ort/11/Gutenberg/etext94/lesms10.txt.gz`)

[2] Stuart J. Inglis. *Planets, stars, and galaxies.* John Wiley & Sons, Inc. 1976.

§ E.6 The names of the notes

Let's start at the very beginning
Kittisak Tiyapan

Let's start at the very beginning,
avery good place to start.
When we read, we begin with A, B, C.
When we sing, we begin with Do, Re, Me.
from a song in the movie *Sound of Music.*

The following text‡, which is written in Latin, leads us back to the origin of the names of musical notes which is the foundation of the familiar tonal music. It mentions John the Baptist who preached at around AD 27 on the bank of Jordan. His two demands were repentance and baptism. One of those people that he baptised was Christ himself. His denouncement of the marriage of Herod Antipas, son of Herod the Great who was a king at the time that Christ was born, with Herodias led first to his imprisonment and later on, to his beheading.

Ut queant laxis
Resonare fibris
Mira gestorum
Famuli tuorum
Sol luce polluti
Labii reatum
Sancte Joannes.

(*mira gestorum*, of wonderful messengers; *famuli*, of a servant (of God); *tuorum*, of visions; *sol luce polluti*, the sun of a sinful person; *labii reatum*, lips of condemnation; *sancte Joannes*, the holy John) Which could be explained as,

As an ode to
resonate (his name) far and wide.
He, of miraculous messengers,
of a servant, and of visions.
The sun of the sinned,
lips of condemnation,
oh! the holy John.

Ut and *Sa* were later on changed to *Do* and *Ti* respectively, probably to make vocalisation easier, so that they finally became the Do, Re, Mi, Fa, So, La, and Ti, which we use nowaday. The pronunciation of the names of these notes is found in a song in the movie the Sound of Music.

Doe a deer, a female deer
Ray a drop of golden sun
Me a name I call myself
Far a long long way to run
Sol a needle pulling thread
La a note to follow Sol
Tea a drink with jam and bread.

However, these names convey a sense of relative pitches with them and they are only used when dealing with music which has tonality. For atonal music, they are replaced by a set of more neutral names, that is A, B, C, D, E, F, and G, for La, Ti, Do, Re, Mi, Fa, and So, respectively. These first seven letters of alphabet represent the absolute pitches rather than the relative pitches. In atonal music, there are neither keys nor chords so that the music drifts about instead of following the structure of a key. Unlike in tonal music where each note has got a specific function different from all others, any note in atonal music is treated as being of the same importance. Further details of this may be found in books on music theory and modern music theory, for tonal and atonal music respectively.

The nature of the two poems, though written centuries apart both works have one thing in common, firstly [sic]*acrostic* where first word of each line forms another group of words, in this case the name of the seven notes. Writing styles also show much similarity between the two poems of totally different purposes. *Assonance*, in other word *rhyming*, can be seen in the first poem (*laxis–fibris*, *gestorum–tuorum–reatum*, and *famuli–poluti*), as well as in the second one (*sun–run*, *thread–bread*). *Alliteration*, the occurrence of the same sound at the beginning of several words, can be seen in the repetition of the *d* sound in *doe*, *deer*, and *drop*. *Metaphor*, the saying of something as being another thing, can be found in the first poem where the author uses the sun (*sol*) when he intends to mean John (*Joannes*). In addition, the second poem uses *pun* to link the name of each note to a meaningful word, except for *La* of which, understandably, a *homophone* is very hard to find.

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‡ from a leaflet obtained from the Holy Name Church, Oxford Road, Manchester, U.K. 1995.

§ E.7 To be unkempt

To be unkempt
Kittisak Tiyapan

A few years ago, two words were brought up to my mind by a very respected person [2] whose profound knowledge in the Englisn language, in my point of view, can hardly find equals. These are the words *unkempt* and *dishevelled*. Both has got a similar meaning. To be *unkempt* is to go around with your hair not combed. It means having untidy hair in milder a degree than having your hair *dishevelled*, which is extremely disorganised.

These two words are very interesting for the reason that at a first glance both appear to be prefixed words, that is *un* + *kempt* and *dis* + *hevelled*. This implies that it should be fine to make the sentences "the groom is both well dressed and kempt today", and "before going to school, hevel your hair first!" (for example, from a mother to her son). But unfortunately there is no such verb "to hevel" in English, neither is there an adjective "kempt". In fact, after a little effort of searching we can find that neither of them is associated to any other english word. Let us take a look into each of these, one by one.

Let us look at the word *unkempt* first. This word can be traced back to a Latin word *incomptus* meaning "unadorned" or "untrimmed" [1, 3]. Similar words include, *úkembdr* in Old Norse, *cemban* in Old English which means "to comb", *kembian* in Old Saxon, *kemben*, *chempen* in Old High German, *kemba* in Old Norse, *ungemembet* in Middle High German, *kämmen* in German, *kambjan*, *kambaz* in Germanic, and *incontos* in Italian which means "unadorned".

After that let us look at the word *dishevelled*. The Latin word *capillus* (from which came *capillary*) means "the hair of the head" gave rise to an Old French word *chevel*. *Chevel* (Old French) can be prefixed to give *descheveler* with its Past Participle *deschevelé*, which in turn produced *dischevel*, *dischevelee* of late Middle English (Chaucer uses *dischevely*). The Past Participle form *dischevele* in Middle English means "with hair in disorder" [4].

References

[1] *Oxford dictionary of English Etymology.* Oxford. 1966.

[2] Amnad Khitapanna. private communication.

[3] C. T. Lewis and C. Short. *A Latin dictionary.* Oxford University Press. 1951.

[4] A. L. Mayhew and W. W. Skeat. *Concise dictionary of Middle English.* Oxford University Press Warehouse.

§ E.8 Critical probability of 2-d tessellation

Critical probability and other properties of 2-d tessellation†

Kittisak Tiyapan‡ G. A. Davies§ David J. Bell‡

2^{nd} September 1996

Abstract

The critical probabilities of Voronoi tessellation and some other uniform lattices, including triangular, square, honey-comb, and Kagomé lattices, are identified by mean of an algorithm developed. The properties of these uniform lattices agree with existing well known results. This work provides the bond critical probability for Voronoi tessellation.

Introduction

A tessellation is an aggregate of cells that cover the space without overlapping. A Voronoi polygon is also known as a Dirichlet polygon, a Wigner-Seitz polygon, a Theissen polygon, a Blum's transform, an S polygon, a cell model, a plant polygon, Wirkkungsbereich, etc.

Definition 1. *Let* Φ *be a distribution of a countable set of nuclei* $\{x_i\}$ *in* R^d*, and let* $x_1, x_2, x_3, \ldots$ *be the coordinates of the nuclei. Then, the Voronoi region is*

$$\Pi_i = \{x | d(x, x_i) < d(x, x_j) \forall j \neq i\}$$

where $d(x, y)$ *is the Euclidean distance between* x *and* y*.*

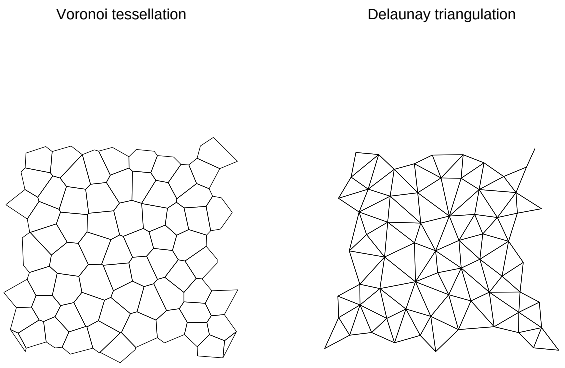


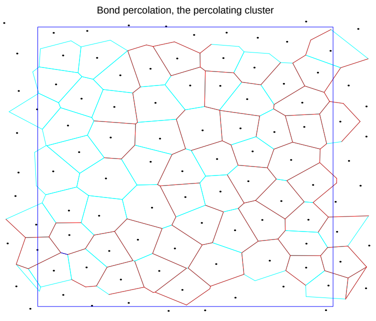
Figure 1.. Duality of Voronoi tessellation and delaunay triangles.

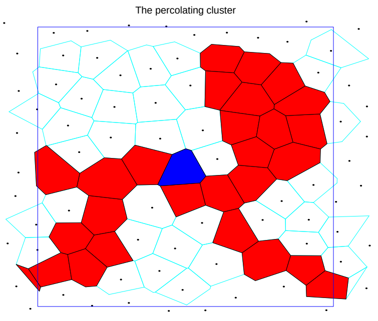
The number of Voronoi points (nuclei) and edges are both $\mathcal{O}(\backslash)$ where $n \in N$, and the number of Delaunay triangles and edges are also $\mathcal{O}(\backslash)$ (Ahuja and Schachter, 1983).

A bond-problem of any lattice L can be translated into a site-problem on L^c, its covering lattice constructed (Shante and Kirkpatrick, 1971) by the following procedure,

1 . Replacing each β_i by α_i^c.

2 . Creating β_i^c by the rule that $\forall \alpha_i^c, \alpha_j^c \in L^c$, α_i^c and α_j^c are connected by a bond of L^c if and only if their corresponding β_i and $\beta_j \in L$ have a common terminal atom of L.





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† This pape was presented at the MTNS96 conference held in St. Louis, U.S., June 1996, by the first author.

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(a) *Example of a bond percolation* (b) *Example of a site percolation*

Figure 2 *Example of* (a) *a bond and* (b) *a site percolation.*

A list of applications can be summarised into two groups according to the two main groups. *Bond percolation* finds application in hydrology (movement of water in dam structure, intrusion of sea water in coastal areas, filter beds, *etc*), petroleum engineering (petroleum and natural gas production, exploration, logging, *etc*), chemical engineering (heterogeneous catalysis, flow through packed beds, gel permeation chromatography, porous polymer films used in separation processes, biological membranes, inorganic membranes, *etc*), medicine and biomedical engineering (biological membranes, biological filters, flow of blood and other body fluids, electro-osmosis, *etc*), electrochemical engineering (porous electrodes, permeable and semipermeable diaphragms for electrolytic cells, *etc*), and in communication (performance of communication networks with blockage). *Site percolation* finds applications in permeation through filtration membranes (Bell *et al*, 1995), sieve blinding (Wilkinson and Davies, 1989), membrane fouling, (for example, of the Anotec (ANOPORE) microfiltration membrane) and in the form of pore clogging, effect of back-flushing and crossflow microfiltration have been studied and close agreement with experiments was obtained.

At and above the *critical probability* p_c (Shante and Kirkpatrick, 1971) a percolating cluster, a cluster that spans infinite length, occurs. A mathematical definition of p_c can be found in Bousquet-mélou (1996). The exact value of p_c for some of the uniform lattices can be found by the method of series expansion (Onody and Neves, 1992; De'Bell and Essam, 1983). But for a Voronoi lattice, at this moment, there is no deterministic method. The only possible way is by doing simulations on the lattice as has been done here.

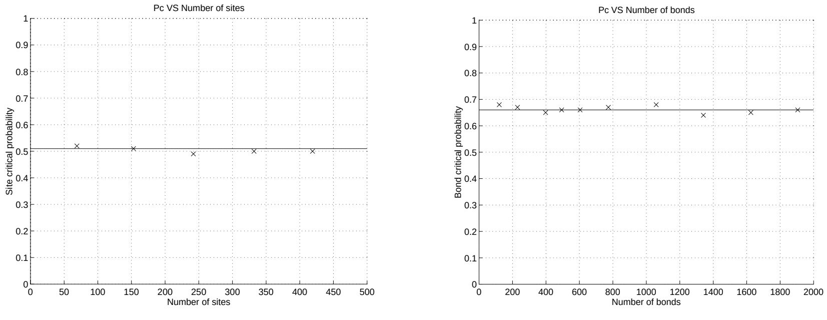
Results

Simulations are made by a developed algorithm (Tiyapan, 1995) based on Monte Carlo method. The results are best presented graphically.

Here $p^s_{c,\text{avg}}$ and $p^b_{c,\text{avg}}$ are the critical probability of site, and bond percolation respectively, averaged over a reasonably large amount of simulation; $n(i)$ is the number of clusters which have i members, where $i \in I$; n is the total number of elements, that is sites for a site problem and bonds for a bond problem; $n(i)_{\max}$ is the size of the largest cluster at a specified p; and p is the ratio between the number of blocked elements and the total number of elements. The coordination numbers for a Voronoi lattice is assumed to be the number of neighbours of each element, averaged over every element within the network. To make clearer a pictorial demonstration of the result, in most of the pictures only the first cluster which percolates is shown.

Discussion

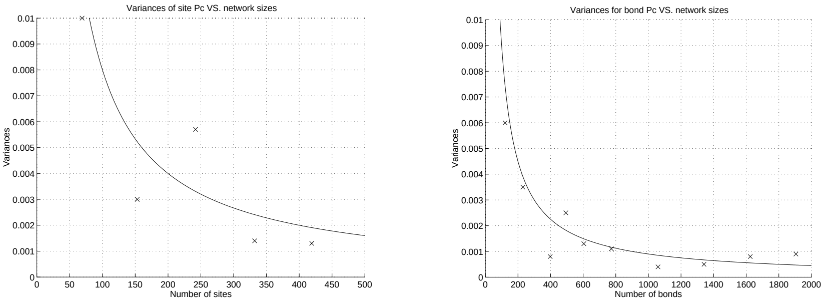
From Figure 3 (a) and (b) it is clear that the critical probability is independent of network size. Thus p_c is a property that is intrinsic for each type of network, and further studies show that it differs from one type of network to another. The idea of infinite cluster is also confirmed.



(a) (b)

Figure 3 (a) $p^s_{c,avg}$*'s plotted against number of sites, (b)* $p^b_{c,avg}$*'s plotted against number of bonds*

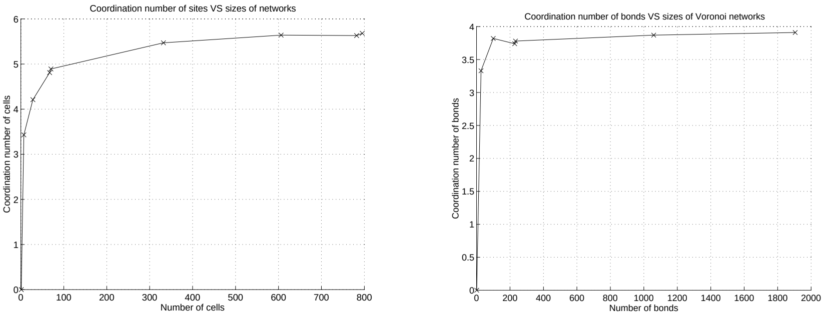
Figure 4 (a) and (b) show that the value of p_c's can be accurately obtained by either doing one simulation on a very large network, or by doing many simulations on a smaller network, since they show that the variance of the results reduces with increasing sizes.



(a) (b)

Figure 4 (a) $\sigma^2_{p^s_c}$*'s, and* (b) $\sigma^2_{p^b_c}$*'s, plotted against sizes of networks.*

Coordination numbers (mean contact numbers) of sites and bonds of Voronoi networks are plotted against network sizes in Figure 5 (a) and (b) respectively. The values seem to be approaching, but never reaching, 6 and 4 for site and bond problems respectively.



(a) (b)

Figure 5 (a) *Coordination number of sites, and* (b) *Coordination number of bonds, plotted against network sizes.*

Figure 6 shows summary of p_c's obtained from all simulations.

Figure 6 [The files for these pictures are lost.] p_c*'s obtained for (a) square lattice,* 220 *bonds,* $p^b_c = 0.53$, $p^b_{c,avg} = 0.47$*; (b) triangular lattice,* 305 *bonds,* $p^b_c = 0.33$, $p^b_{c,avg} = 0.34$*; (c) honeycomb lattice,* 111 *bonds,* $p^b_c = 0.65$, $p^b_{c,avg} = 0.64$*; (d) Kagomé lattice,* 174 *bonds,* $p^b_c = 0.55$, $p^b_{c,avg} = 0.52$*; (e) [399 nuclei, 240 cells,* $p_c = 0.47$*, limit* $= 0.04$*] site problem for Voronoi lattice,* $p^s_{c,avg} = 0.51$*; and (f) [400 nuclei, 324 cells, 1044 bonds, bond percolation, limit* $= 0.04$, $p_c = 0.65$*] bond problem for Voronoi lattice,* $p^b_{c,avg} = 0.66$.

Conclusion

Critical probability is a value which is intrinsic to each type of networks. The value for each uniform or random lattices is constant and does not depend on the size of network.

[All the simulation] result shown in this paper was done by MATLAB running on UNIX workstations. The program used for the generation of Voronoi lattices was adapted from Jafferali (1995). A Kagomé lattice can be generated from either a triangular lattice or a honey-comb lattice (Tiyapan, 1995).

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§ E.9 Some properties of stochastic optimal control

Some properties of stochastic optimal control¶

Kittisak Tiyapan†

17th July 1996

Abstract

Model-based stochastic optimal controller has been studied extensively both by theoretical approach and by simulation. This work provides more result from simulations, in comparison with the existing theory.

Introduction

Performance function of a system can sometime either be unknown to us or difficult to find. In practice, this can be found in various disciplines, for example performance of a plant, profitability of an investment scheme, *etc.* Sometime it is also important to know the true shape of the function. But in many cases, what is even more important for practically is how to keep a plant running at the highest performance possible, or to keep doing a business at the most profitable point at all time.

In this work, control of plants with unknown performance functions subjected to noises are simulated using a stochastic controller, namely extremum controller. The controller uses model-based hill-climbing technique and recursive least square algorithm. Theory, as well as applications of various control schemes, can be found in a reference (Wellstead and Zarrop, 1991). Convergence properties, for example convergence with probability one, possible convergence points, and asymptotic behaviour of recursive algorithms have geen made possible by relating these algorithms to deterministic differential equations (Ljung, 1977). One example is the analysis of convergence properties given in (Zarrop and Rommens, 1993) for a multi-input adaptive extremum controller.

Statement of the problem

The model chosen is

$$y = au^2 + bu + c, \tag{1}$$

where y is the performance index to be maximised and u is the adjustable factor (control input).

If $a < 0$, then (1) represents a hill (not a valley) with a summit at

$$u = -\frac{b}{2a}. \tag{2}$$

The performance of the plant used in simulation is given by

$$y = g(u) + \text{noise}, \tag{3}$$

where the noise represents sensor noise.

In the simulations performed, several shapes of the performance function are chosen, namely quadratic, sinusoidal, third order polynomial, discontinuous functions, asymmetric functions, *etc.*

Algorithm

At points closer to the optimal point the gradient lessens and the algorithm using a hill-climbing technique slows down. The algorithm used utilise a *dither* signal to avoid this slowing-down effect. The algorithm thus can be briefly described by

$$u(t) = -\frac{\hat{b}(t)}{2\hat{a}(t)} + v(t), \tag{4}$$

$$\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I}_3 - \frac{\mathbf{x}(t)\mathbf{x}^T(t)\mathbf{P}(t-1)}{\lambda + \mathbf{x}^T(t)\mathbf{P}(t-1)\mathbf{x}(t)}\right\}, \tag{5}$$

$$\hat{\theta}(t)\hat{\theta}(t-1) + \mathbf{P}(t)\mathbf{x}(t)\left\{y(t) - \mathbf{x}^T(t)\hat{\theta}(t-1)\right\}, \tag{6}$$

where

$$\hat{\theta}(t) = \left[\hat{a}(t), \hat{b}(t), ch(t)\right]^T,$$
$$\mathbf{x}(t) = \left[u^2(t-1), u(t-1), 1\right]^T.$$

¶ This work was done while the author was studying at the University of Manchester Institute of Science and Technology, Manchester, U.K.

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Both the noise affecting the plant and the dither signal are assumed to be zero-mean, uncorrelated processes.

Results

The results can be best illustrated using pictures obtained from simulations.

§ E.10 Fractals in traffic control

Fractals in traffic control

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Fractals as well as *geometrical structures* of networks have been well-known in many areas, for examples the dispersion of fluid in porous media [2], particle fouling on filters [1], and analysis of sieve blinding [5]. In these applications they have played an important part in modelling, thus enabling both process control and product quality control. *Percolation theory* has been studied extensively since 1957 and there have been a numerous number of papers on the subject as well as on its applications. Modern *geometry* provides a power tool for applications ranging from space-time cosmology (*eg penrose diagram*) to the internal structure of quasicrystalline alloys [4]. Some list of the earlier literatures can be found in [3]. Furthermore, both percolation theory and fractals have been found to be related to each other [1]. The application of either theory to a traffic networks has not yet been proposed despise the obvious similarity between random networks found in nature and the traffic networks, which is also random. So far, models used in the study of traffics are divided roughly into two types with respect to the two approaches, namely microstructure approach which studies the behaviour of individual cars and interactions between them and macrostructure approach which studies the overall picture of the traffic, that is the traffic flow. The latter approach is closely related to fluid flow. In this project [the aim is to study] the possibility, the method, and the feasibility of the application of both fractals theory and percolation theory to the modelling of traffic networks and, if possible, to extend the result to traffic control and to congestion control.

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§ E.11 Distributed parameter systems

Distributed parameter systems

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Let $A, B, \ldots, F$ be functions of (x, y). Consider the linear PD operator $L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$. A problem is to find a function $\varphi(x, y)$ such that $L\varphi = G$ in the region R of the x, y-plane, such that certain conditions are satisfied on a part of the boundary of R. The form is determined by the *discriminant* $B^2 - AC$. The equation is called *hyperbolic* when $B^2 - AC > 0$, is called *parabolic* when $B^2 - AC = 0$, and is called *elliptic* when $B^2 - AC < 0$. A distributed parameter system (DPS) is a dynamical mathematical model represented by partial differential equations. It is called *distributed* because the state of the system depends on spatial coordinates as well as on input variables. Some examples [1] of DPS are the heat equation $\rho C\frac{\partial y}{\partial t} = \mathrm{div}(K \,\mathrm{grad}\, y) + \rho F$ describing heat conduction systems, the two-dimensional wave equation $-\omega \Delta y = f$ in $\Omega \times (0, T)$, $\Omega \in R^2$ describing the motion of a transversely vibrating membrane, the evolution of substrate concentration of a one-dimensional membrane $\frac{\partial S}{\partial t} - D\frac{\partial^2 S}{\partial x^2} + \frac{VS}{K+S} = 0$, $x \in (a, b)$, $t \in [0, T]$ describing a *substrate plus enzyme to product* reaction in biochemistry, the linear hyperbolic equation $\frac{\partial l}{\partial t} + \frac{\partial l}{\partial x} = -\mu l$ describing population dynamics with $l(x, t)$the density with respect to age x of a population at time l and $u \geq 0$ the fertility function, and the reaction-diffusion equation $\frac{\partial y_i}{\partial t} = \alpha_i \Delta y_i + f_i(t, y_1, \ldots, y_n; u_1, \ldots, u_m)$ in $\Omega \times (0, T)$, $i = 1, \ldots, n$ where $y_i(x, 0) = y_i^0(x)$, $x \in \Omega$; $\alpha_i > 0$ and $\frac{\partial y_i}{\partial v} = \rho_i(t, y_1, \ldots, y_n; v_1, \ldots, v_p) \in \Gamma \times (0, T)$ used to model multigroup neutron diffusion, compustion theory, and many other physical processes. In this project the systems with mixed components of lumped parameter and distributed parameter are studied, as well as the control of DPS by different methods, for example [4] by strong coupling (singular perturbation), weak coupling (ϵ-coupling), *etc.* Applications to flexible structures, some examples of which are Wrap-Rib antenna [3] and a flexible robot arm [2].

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§ E.12 Vision robots

Vision robots

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Stochastic control theory has proved to be useful for controlling of industrial processes [1]. With dynamic plants, and in order that plants can process without interruption to adjust parameters, it is desirable to have a regulator which can tune its parameters on-line. The PID controller needs to be re-tuned after each change in the operating region or the plant dynamics, which is not always desirable. One solution has been to use a regulator which has self-tuning property [2] where the plants can be left unattended for a long time because the identification tasks being done off-line instead of on-line. The time constants for these systems are normally large. An example of these applications is the control of a crushing plant of ore used in mineral processing. Probably due to the limitted microprocessor speed and the natural degree of unreliability [that] exists in parameter estimation, the appxlication to areas which require a much faster response, for example vision robots, has seldom been looked at. But [as a result of] the decrease in hardware price, the increase in efficiency of computers, as well as better mathematical tools [4], it may be possible to find more reliable applications in these areas [3]. This project finds ways to incorporate stochastic control schemes into prospective applications, one of which is vision robots. The study of different algorithms and the hybrid among them is one of the objectives. Though the work is concentrated on doing simulations on computer, the relisation of the results on real robots is also one of the ultimate goals.

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§ E.13 Singular perturbation

Singular perturbation

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Singular perturbation technique can be used to reduce the order of the system by neglecting fast phenomena in $\dot{x} = f(x,z,u,t,\mu)$, $\mu\dot{z} = g(x,z,u,t,\mu)$, $\mu > 0$ by letting $\mu = 0$ to obtain a *reduced* model $\bar{x} = f[\bar{x},\phi(\bar{x},\bar{u},t),\bar{u},t,0] \equiv \bar{f}(\bar{x},\bar{u},t)$ where $\bar{z} = \phi(\bar{x},\bar{u},t)$ [6]. An open problem regarding the extent of order reduction has been raised [2] on how far can the size of $\mathbf{A}$ in a physical system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{r}$ be reduced for the result to remain accurate as a representation to a process. Some of the results found in literature are sufficient condition for preservation of controllability under perturbations of a class of nonlinear control systems [4], stability robustness of a state space description for perturbed linear systems of the form $\dot{x} = A(k)x$, $A(k) = A_0 + \sum_{i=1}^{p} f_i(k)A_i$ where $k \in R^m$ is a perturbation vector [3], the asymptotic behaviour of solutions of a Volterra integrodifferential system of the form $x' = f(t) + A(t)x(t) + \int_0^t B(t,s)x(s)ds + (gx)(t)$, $x(0) = x_0$ [5] where gx which is represented by $g\varphi$ is a small nonlinear functional, for example $g\varphi(t) = \int_0^t c(t,s)\varphi^2(s)ds$ or $\varphi t \int_0^t c(t,s)\varphi(s)ds$ which occur in reactor dynamics. Some of the applications cited in or worked out in existing literature are optimal thrust control of guidance missiles [1], singular perturbation methods used when small time delay is neglected in order to obtained a solvable problem, the use of singular perturbation methods as a tool for asymptotic analysis and design of optimal control of distributed parameter systems [6], and the use of singular perturbation approach in both the variable structure systems and the cheap control problem [7]. In this project singular perturbation is being studied with the emphasis on application to distributed parameter systems and time delay systems. Also, possible new applications to physical systems is being investigated.

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§ E.14 Object-location using Extremum Control

On an algorithm for object-location problem

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5th October 1996

Abstract

Self-tuning extremum control is useful when dealing with systems with performance index measured in a noisy environment. This work studies the approach using the model-based hill-climbing technique. The recursive least square (RLS) algorithm is used on a quadratic hill whose height represents the location of an object.

1. Introduction

The term *extremum control* (Wellstead and Zarrop, 1991) is a synonym for optimisation. It means making a process operate at the optimal point at all times. The optimal point is normally obtained by extremising a performance index. A model-based algorithm is an algorithm where the output from a reference model, as well as measured input/output data, are used to control system parameters. In self-tuning extremum control the inputs and output of the system considered are measured and used to adjust parameters of the model, in order to move toward the point where performance index is maximised, often by mean of a hill-climbing technique.

Hill-climbing is a very useful technique. We can imagine a performance function as being a hill. The performance index is represented by height of the hill [at] the point considered. If we compare this performance index with those of the points in our close neighbourhood we will know which direction to move to in order to increase our performance index. The process is carried out recursively until we reach the top of the hill where no neighbouring point leads to any improvement in performance.

Depending on how much knowledge of the hill we have at the beginning, we may be able to fix some of the parameters without any effect to the final result. For example, if our model is a quadratic one, we may be able to fix the coefficient of the square term. In so doing, our estimated hill will still give the correct information of the position of the hill top. This is enough for most practical situations where we only need to know the position of the hill top, in this case the location of our object, without the need to know the actual height of the hill itself. In situations where the height of the hill represents a performance index, we do not need to know the value of the performance index in order to find the optimal operating point.

Self-tuning control (Wellstead and Zarrop, 1991) is closely related to adaptive control. It is useful when dealing with systems with performance index measured in a noisy environment. Here the input and output of the system are measured and used to adjust model parameters. Using recursive least squares algorithm, it is possible to extend the approach to multi-input case (Zarrop and Rommens, 1993) and hence to an object-location problem.

In the neighbourhood of extremum a quadratic model is a good representation of the hill top. If the hill top is smooth, this model represents the first few terms of a Taylor's series for the performance function. Because the recursive least-square algorithm uses the difference in estimated value compared with the actual value, the smoothness of the hill-top results in the slowness of convergence of the algorithm when approaching the summit. Here an algorithm slows down considerably (goes to sleep) unless a *dither signal*, normally a small signal with zero-mean, is added to the control input.

In this paper, various algorithms are investigated. These includes quadratic, reduced-parameter, and multi-input models. Simulations are conducted for these model, using MATLAB, and a two-input algorithm is applied to an object location problem.

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The organisation of this paper is as follows: Section 2 describes the three models to be used, Appendix A gives the respective algorithms for those models, Section 3 shows simulations using the three algorithms and result obtained, in Subsection 3.4 an object location problem is considered as a possible application of the method, Section 4 discusses the result obtained from simulations.

2. Model description

Assuming the top of the hill smooth and represent it by Eq.1.

$$y = au^2 + bu + c \tag{1}$$

where y is the performance index to be maximised and u is the control input. When $a < 0$, Eq. (1) represents a hill (while $a > 0$, a valley) with a summit at

$$u = -\frac{b}{2a}. \tag{2}$$

Let the true hill be

$$y = g(u) + \text{noise} \tag{3}$$

The number of estimation parameters can be reduced from three to one. Let

$$\Delta y(t) = y(t) - y(t-1) \tag{4}$$

then

$$\Delta y(t) = a\Delta u^2(t-1) + b\Delta u(t-1) \tag{5}$$

Secondly we may fix $a = a_f < 0$ and (5) may be written as

$$y'(t) = \mathbf{x}^T\theta \tag{6}$$

where

$$y'(t) = \Delta y(t) - a_f\Delta u^2(t-1) \tag{7}$$

$$\mathbf{x}(t) = \Delta u(t-1) \tag{8}$$

$$\theta = b \tag{9}$$

Then, based on the model

$$\Delta y(t) = a_f\Delta u^2(t-1) + b\Delta u(t-1) \tag{10}$$

In the case when the control input is a vector (*i.e.* we have a number of adjustable factors that influence performance) (1) can be generalised to

$$y = \mathbf{u}^T A\mathbf{u} + \mathbf{b}^T\mathbf{u} + c \tag{11}$$

where $\mathbf{u}$ is the control vector (of dimension m) and A is a symmetric matrix ($m \times m$). If we estimate all the parameters in this model, we will have $\frac{1}{2}(m+1)(m+2)$ of them.

For an object location problem, the two inputs are both coordinates of the object considered. Data is collected by mean of a small mask of known size, placed randomly first, then according to the hill climbing technique once the mask can detect the object. The recursive least square (RLS) algorithm continues as long as the positions of the mask and the object still overlap, otherwise another random search is resumed. The height of the hill is represented by the area of overlap.

3. Simulations and result

In all simulations, noise is generated as a gaussian random variable with zero mean and adjustable variance. Dither signal is chosen as a zero mean uncorrelated process with uniform distribution.

3.1. Quadratic model

Simulations begin with a quadratic model using Algorithm 1. Several hill shapes are used, these include circular-top, sinusoidal, third order polynomial, fourth order polynomial, sinc function, symmetric piece-wise functions, asymmetric piece-wise function, and pulse shaped function. When hills have more than one extremum, convergence is toward one of them, which may not be the closest one to the starting point. With badly conditioned hill, for example a pulse shaped function, it takes longer to find the summit which, in this case, is any point on top of the pulse. Bias exists with all non-symmetric hills. Smaller dither signal variance results in less bias in the result and, however, it also results in slowness in reaching the target.

The forgetting factor (λ) between 0.95 and 1.0 gives good result. Smaller λ produces unreliability in result, while when $\lambda > 1$ the result does not converge to the true value. When the dither signal variance σ_v^2 is small, the parameters' values estimated scatter around the true value. This can be easier illustrated using a plot between two parameters. As can be expected, when noise variance σ_e^2 increases, uncertainty in estimation also increases. These results are generally true for the simulations to follow as well.

3.2 Reduced parameter model

In cases where we have some knowledge about the nature of our hill from the start, for example whether it is a hill or a valley, we may simplify our operating model by reducing some of the parameters. Algorithm 2 is used in doing the simulation.

Figure 1 *A reduced parameter model with a quadratic hill. Shown are the true hill (top left); an estimating hill at the start, true hill, and an estimated hill (top right); control input against number of steps (bottom left); and the estimated parameter* $\hat{b}$ *during simulation (bottom right).*

3.3. Multi-input model

In the simulation, which uses Algorithm 3, some of the components in matrix $\hat{A}$ are fixed. The result using this model helps dealing with the problem in the next section.

Figure 2 *A two-input model. Shown are the true hill (top left); an estimating hill at the start (top middle); an estimated hill (top right); the previous three combined together (bottom left); estimated parameter b's for both inputs during simulation (bottom middle and right).*

3.4. Object location problem

Algorithm 4 is used for the simulation. Here the hill is badly conditioned, that is there is no sloping part at points away from the object to give the indication of direction. Therefore random search is adopted which allows proximity search before a more precise locating using the RLS algorithm.

Figure 3 *Object-location when object is shaped like an O-letter. Shown are a picture of the object (top left); the true hill (top left); and the two coordinates of mid-point of estimated image (bottom). Here we have* $\sigma^2_{noise} = 0.05$, $\sigma_{rough\ dither^2} = 3$, *and* $\sigma_{fine\ dither^2} = 0.03$

Figure 4 *Snapshots of ogjects at various time steps.*

4. Discussions

Simulations show that the speed of convergence is improved when we use a dither signal with a large variance. The symmetrical nature of the hill determines the choice of dither signal. For an improved estimate when the hill is asymmetric, a smaller dither signal is needed. Smaller dither signal results in slower convergence. Therefore, with symmetric performance functions, a trade-off between the speed and accuracy of estimation is needed.

With hills without slope, a random search is incorporated to the existing algorithm which produces a better result as, for example in object location problem. Improvements are still needed in order to deal with the situation where the estimates loses touch with the actual object, as can be seen in Figure 5. There still remain a problem regarding precision when the hill is asymmetric. This problem can be solved, providing that the environment is not too noisy, by incorporating a subroutine withing the algorithm that switches the dither signal variance to lower values once the position of the object has been estimated.

It is also interesting to note that, taking a step further by incorporating a subroutine which can change the shape of the mask used for detection, and by taking into account the percentage of the area of the mask taken up by the object, the object locator will be able to tell the shape of the object in addition to the location found.

5. Conclusion

Algorithm for finding position of an object is proposed, and simulations done, based on the idea of self-tuning control. To be of further use in practice, more simulations must be done to obtain an optimum algorithm for the purpose. One prospect of the study is in the area of vision robot and object tracking.

6. Acknowledgements

The author would like to thank Dr. M. B. Zarrop for his kind advices for the designing exercise *Object-location using Extremum Control* in an M.Sc. course in Control and Information Technology at the University of Manchester Institute of Science and Technology, which this work is related to.

A. Algorithms

These are four algorithms used in Section 3.

Algorithm 1 (*for quadratic model*)

```
S0  $t = 0; \hat{a}(0), \hat{b}(0), \hat{c}(0), \mathbf{P}_{(3\times 3)}(0)$
S1  $u(t) = -\frac{\hat{b}(t)}{2\hat{a}(t)} + v(t)$
S2  $t = t + 1$
    $y(t) = g(u(t-1)) + e(t)$
S3  $\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I}_3 - \frac{\mathbf{x}(t)\mathbf{x}^T(t)\mathbf{P}(t-1)}{\lambda + \mathbf{x}^T(t)\mathbf{P}(t-1)\mathbf{x}(t)}\right\}$
    $\hat{\theta}(t) = \hat{\theta}(t-1) + \mathbf{P}(t)\mathbf{x}(t)\left\{y(t) - \mathbf{x}^T(t)\hat{\theta}(t-1)\right\}$
S4  goto Step 1
```

```
NB
    $\hat{\theta}(t) = \left[\hat{a}(t), \hat{b}(t), \hat{c}(t)\right]^T$
    $\mathbf{x}(t) = \left[u^2(t-1), u(t-1), 1\right]^T$
```

Algorithm 2 (*for reduced parameter model*)

```
S0  $t = 1; a_f = -7, \hat{b}(0), \hat{b}(1), P(1)$
S1  $u(t) = -\frac{\hat{b}(t)}{2a_f} + v(t)$
S2  $t = t + 1$
    $y'(t) = bx(t) + e(t)$
S3  $P(t) = \frac{1}{\lambda}P(t-1)\left\{1 - \frac{x^2(t)P(t-1)}{\lambda + x^2(t)P(t-1)}\right\}$
    $\hat{b}(t-1) + P(t)x(t)\left\{y'(t) - \hat{b}(t-1)x(t)\right\}$
S4  goto Step 1
NB
    $x(t) = \Delta u(t-1) = u(t-1) - u(t-2)$
```

Algorithm 3 (*for the multi-input model*)

```
S0  $t = 1; \hat{a}_1(0), \hat{a}_2(0), \hat{b}_1(0), \hat{b}_2(0), \hat{c}_1(0), \hat{c}_2(0), \mathbf{P}_{6\times 6}(0)$
S1  $p_1(t) = \frac{\hat{b}_1(t)}{2\hat{a}_1} + v_1(t)$
    $q_2(t) = \frac{\hat{b}_2(t)}{2\hat{a}_2} + v_2(t)$
S2  $t = t + 1$
    $y(t) = g(\mathbf{x}(t)) + e(t)$
S3  $\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I}_6 - \frac{\mathbf{x}(t)\mathbf{x}^T(t)\mathbf{P}(t-1)}{\lambda + \mathbf{x}^T(t)P(t-1)\mathbf{x}(t)}\right\}$
    $\hat{\theta}(t) = \hat{\theta}(t-1) + \mathbf{P}(t)\mathbf{x}(t)\left\{y(t) - \mathbf{x}^T(t)\hat{\theta}(t-1)\right\}$
S4  goto Step 1
NB
    $\mathbf{x}(t) = \left[p^2(t-1), q^2(t-1), p(t-1), q(t-1), 1, 1\right]^T$
    $\hat{\theta}(t) = \left[\hat{a}_1(t), \hat{a}_2(t), \hat{b}_1(t), \hat{b}_2(t), \hat{c}_1(t), \hat{c}_2(t)\right]^T$
```

Algorithm 4 (*for object location problem*)

```
S0  $t = 0; \hat{a}_1(0), \hat{a}_2(0), \hat{b}_1(0), \hat{b}_2(0), \hat{c}_1(0), \hat{c}_2(0), \mathbf{P}_{6\times 6}(0)$
S1  $p(t) = -\frac{\hat{b}_1}{2\hat{a}_1} + v(t)$
    $q(t) = -\frac{\hat{b}_2}{2\hat{a}_2} + v(t)$
S2  $(p, q)$ is mid-point of the mask
    $y =$ overlapped pixels
S3  if $y > 1$ then
        $t = t + 1$
        $\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I} - \frac{\mathbf{x}(t)\mathbf{x}^T(t)\mathbf{P}(t-1)}{\lambda + \mathbf{x}^T(t)\mathbf{P}(t-1)\mathbf{x}(t)}\right\}$
        $\hat{\theta}(t) = \hat{\theta}(t-1) + \mathbf{P}(t)\mathbf{x}(t)\left\{y(t) - \mathbf{x}^T(t)\hat{\theta}(t-1)\right\}$
     else
        random search for $p$ and $q$
        $\hat{b}_1 = -2a_1p$
        $\hat{b}_2 = -2a_2q$
S4  goto Step 1
NB
    $\mathbf{x}(t) = \left[p^2(t-1), q^2(t-1), p(t-1), q(t-1), 1, 1\right]$
    $\hat{\theta}(t) = \left[\hat{a}_1(t), \hat{a}_2(t), \hat{b}_1(t), \hat{b}_2(t), \hat{c}_1(t), \hat{c}_2(t)\right]^T$
```

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§ E.15 On pragmatists and idealists

On pragmatists and idealists

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21^{st} October 1996

There are two important qualities which together help making great men of the world. They are pragmatism and idealism. Idealism requires that a person has got an idea in which he believes to be a good cause in doing something. An idealist sticks to his ideal and his action is influenced by it. On the other hand, a pragmatist is a practical person who takes into account methods which are available at that time, as well as the constraints involved, into account. Pragmatism implies that one has got to use tricks sometimes. Idealic without pragmatic is futile, while pragmatic without idealic is stupidity. To put in another words, pragmatism alone makes a man without a heart, while idealism alone makes a man without a head.

Take for an example, the play *Julius Caesar* by William Shakespeare. The play depicts the returning of Julius Caesar to Rome after his victories in far lands. He was murdered by a group of conspirators which was led by Decius Brutus, but which was designed by Cassius. In this play Shakespeare has shown us three different personations, the first one represents a pragmatist, the second one an imperfect combination of the two qualities, and the third one a perfectly balanced person who has both a pragmatic and an idealic quality.

The first one is Cassius who, with his fluent words, succeed in inducing other people, including Brutus, to help murdering Caesar. He hates Caesar, and has no qualm in killing him. His character was described by Caesar himself when he speaks to Mark Antony,

I do not know the man I should avoid
So soon as that spare Cassius. He reads much;
He is a great observer and he looks
Quite through the deeds of men: he loves no plays,
As thou dost, Antony; he hears no music;

Julius Caesar, Act 1 Scene 2.

Cassius is being described here as a man with knowledge, because he reads much, but who is without inner good qualities necessary to make a man, because he neither like plays nor like music. Therefore he is a stereotype of our previous definition, that is *with head but without heart.* Caesar mentioned here Mark Antony as the opposite side of Cassius, which he is, as we shall see afterward. Here, also, Caesar seems to know beforehand that there will be trouble coming from the direction of Cassius, though he does not know how it will happen.

The next character is Brutus, who also loves Caesar, but either because he succumbs to his own jealousy in Caesar or because he is led towards [that] direction by the clever Cassius, has decided to take part in the bloody business and stabs Caesar in the back. This statement can be seen in the following lines taken from Act 1 Scene 2 of the play, where he says to Cassius.

I would not, Cassius; yet I love him well.
But wherefore do you hold me here so long?

Julius Caesar, Act 1 Scene 2.

Antony is portrayed by Shakespeare to possess a perfect harmony between the two qualities. As a leader, he is both an idealic and a pragmatic at the same time. He has much love for both Caesar and Rome. After Caesar has been killed by the conspirators Antony flees to his own house and hides there. Realising that he will be the next target of attack and knowing that the force from the other side is too great for him to resist, he sents a servant to prostrate himself before Brutus. This is what happens.

Thus, Brutus, did my master bid me kneel:
Thus did Mark Antony bid me fall down;
And, being prostrate, thus he bade me say:
Brutus is noble, wise, valiant, and honest;
Caesar was mighty, bold, royal, and loving:
Say I love Brutus, and I honour him;
Say I fear'd Caesar, honour'd him and loved him.

Julius Caesar, Act 3 Scene 1.

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Then without a word of lie he joins the group of his enemy to the market-place to meet a crowd of roman citizens. At the market-place Brutus gives a speech. He tells the people of Rome that in killing Caesar he has saved Rome from ruin because Caesar was ambitious, and that although he loves Caesar much he loves his country even more. The people believe what Brutus says. After that Brutus Antony to come upon the stage to talk to the people also. This Antony does, and he gives such a good speech that at the end of it all the romans turn against the conspirators and demand vengeance for the death of Caesar. His speech is so tactful that it conveys different meanings to two groups of people who are listening to it at the same time, that is the conspirators and the people. Antony wants to rouse the people, who by this time have already agreed with what Brutus has said, to go against him. To do this he needs to build up the emotion of the mob without letting his enemy realise what he is at, otherwise they can stop him at any time and all his effort will be in vain.

> Friends, Romans, countrymen, lend me your ears;
> I come to bury Caesar, not to praise him.
> The evil that men do lives after them;
> The good is oft interred with their bones;
> So let it be with Caesar. The noble Brutus
> Hath told you Caesar was ambitious:
> If it were so, it was a grievous fault,
> And grievously hath Caesar answer'd it.
> Here, under leave of Brutus and the rest–
> For Brutus is an honourable man;
> So are they all, all honourable men–
> Come I to speak in Caesar's funeral.
> He was my friend, faithful and just to me:
> But Brutus says he was ambitious;
> And Brutus is an honourable man.
> He hath brought many captives home to Rome
> Whose ransoms did the general coffers fill:
> Did this in Caesar seem ambitious?
> When that the poor have cried, Caesar hath wept:
> Ambition should be made of sterner stuff:
> Yet Brutus says he was ambitious;
> And Brutus is an honourable man.
> You all did see that on the Lupercal
> I trice presented him a kingly crown,
> Which he did thrice refuse: was this ambition?
> Yet Brutus says he was ambitious;
> And, sure, he is an honourable man.
> I speak not to disprove what Brutus spoke,
> But here I am to speak what I do know.
> You all did love him once, not without cause:
> What cause withholds you then, to mourn for him?
> O judgement! thou art fled to brutish beasts,
> And men have lost their reason. Bear with me;
> My heart is in the coffin there with Caesar,
> And I must pause till it come back to me.
>
> *Julius Caesar, Act 3 Scene 2.*

In it he says all the time that the traitors are all honourable men. By *honourable men* it neither imply that they are men of virtue or that their deed is right or pardonable. Talking about their claim that Caesar was an ambitious man, he gives several examples of the good things which Caesar has done for Rome, for example the ransom received in exchange for captivated enemies, the love of Caesar for the people including the poors, and the rejection of the crown which was being offered to him by Antony himself at previous public gatherings. Through these carefully selected examples he has shown to the romans that if Caesar is ambitious he is so only for the prosperity of Rome and its people. Nor that *ambition* [is] a negative quality, if it does not have anything to do with selfishness. Among its synonyms is the word *aspiration* which definitely implies a positive quality of a person. For the reason given, this speech made by Antony is much studied by students of English literature and is very well known to the rest of the English speaking people. From Figure 1 we can see how Antony managed to change the direction of the emotion of the romans. After that he talks about a will written by Caesar, which he says he is not going to read.

> But here's a parchment with the seal of Caesar;
> I found it in his closet, 'tis his will:
> Let but the commons hear this testament–
> Which, pardon me, I do not mean to read–
> And they would go and kiss dead Caesar's wounds
> And dip their napkins in his sacred blood,
>
> *Julius Caesar, Act 3 Scene 2.*

[Figure 1 used to be here.]

He stresses that it is not his wish to read the will lest the people will mutiny against *the noble men.* The effect which is produced after that can be seen in the steep increase in emotion in Figure 1 as well as an excerpt from the following short dialogues among the citizens, which also helps us conclude that Mark Antony is portrayed in this play to be both an idealist and a pragmatist at the same time.

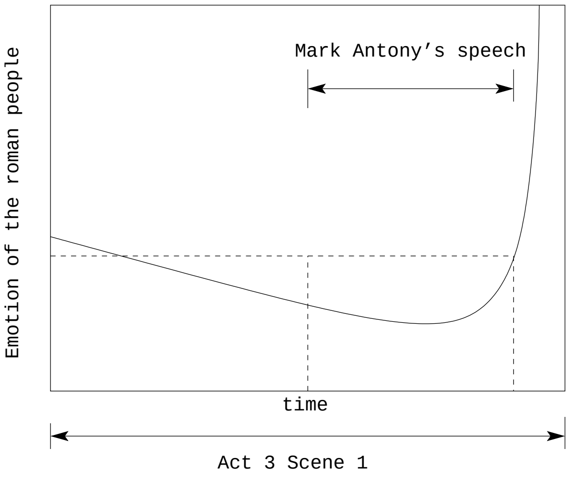

Figure 1. The emotion of the roman people during the speech of Mark Antony.

All We'll mutiny.
First Citizen We'll burn the house of Brutus.
Third Citizen Away, then! come, seek the conspirators.

Julius Caesar, Act 3 Scene 2.

§ E.16 The Morris Worm

The Morris Worm

Kittisak Tiyapan

30th October 1996

In 1988 a piece of 99 lines of code written by a 23 year-old Ph.D. student in Computer Science at Cornell University, Robert T. Morris, had crippled a large number of computers connected to the internet. At 6 pm 2nd November 1988 , a worm was set loose by its author. Within less than five hour's time from the launch, computers across US had to be shut down. At the time when Andy Sudduth at Harvard posted a message saying that *there may be a virus loose on the internet*, the internet was already coming apart. All the computers affected were not able to come up again until all the working worms had been removed from the internet, which was four days later. The program, which has been known afterward as the Morris Worm, had forced more than 6,000 workstations, all of which were VAX and Sun machines, to be shut down. The estimated loss was thought to easily reach several million US dollars. It was claimed by Morris, afterward, that the effect from the code was meant to be much milder but for some small bugs within it. As a result, the whole of the internet community was shaken.

In contrast to a virus which mainly spreads itself via infected disks, a worm goes via the internet. A virus alters a file and waits for the file to be activated by a user. A worm, on the other hand, acts on its own and is far more aggressive than a virus. Take the Morris worm as an example. It protects itself, seeks and stores information it needs, searches for victims and then attacks them, reports its progress, balances the situation and makes decisions accordingly, and even destroys evidences and commits suicide. Moreover, the worm has got its own built in dictionary of words to be used in cracking passwords. To put in plain words, a worm is a killer robot while a virus a parasite.

During the worm attack there were efforts trying to decompile the program in order to find remedy. Among these were those made by the teams at the University of California at Berkeley, MIT, and at Purdue. Within less than 24 hours the team at Berkeley and the team at Purdue had found methods to slow down the spread of the worm. Ironnically, not being less panicked at the fast spread of the worm was its author himself. Morris had contacted his friend at Harvard. They discussed a solution and eventually came up with one. Unfortunately the messages sent by these teams and others to tell about the remedies could not reach some of the affected sites as fast as they should have. This was because of the clogging of the networks. Also, at many of the sites the mail facility was shut down after the discovery that the worm spread via `sendmail`. Thus the message about the remedy were even further delayed.

Since the happening, the internet community has become much more aware of the importance of the security of their networks and systems. Within the same month of which the worm had appeared, an organisation was formed in response to the incident. It was called CERT, which stands for *Computer Emergency Response Team.*

One of the interesting facts is that most of the computer viruses and worms that ever were, came from the acedemics. It is widely believed, at least within community of people who see the importance of security in computer systems, that the cases of these so-called *white-collar* crimes must be much higher than what we have heard. Many of the cases discovered are believed to have been covered up, and dealt with internally, in order to keep the good image which many companies have got, especially financial institutions and government authorities. Others simply goes undetected. The irony in this is that while the money involved increases exponentially in relation to the cases of its counterpart which are known as the *blue-collar* crimes, the risk of being caught is comparatively small.

The field of computer and networks security is one which comprises of two main parties, those who try to prevent on on side, and those who try to infringe on the other. As the technology leaps forward the way it is now, both parties will find it hard already only to keep abreast with new products and to catch up with new methods used by the other party. Though the preventors seem to be better manned and better equipped, it has got a difficulty of having to deal with enemies who are in the dark. Also security does not come at no cost. On top of the cost of hardwares, softwares and personnels, there is also an inevitable trade-off between convenience of the users and the security of the systems. In other words, though in theory it might be true to say that the more secure the system is the better. But in reality it all depends upon the consideration about the costs against the risks involved.

Miscellaneous facts

The Morris worm made use of `finger` command. This is one of the reason why many sites now block inbound `finger`ing requests.

The file `/etc/passwd` is world-readable. The security loop-hole of storing encrypted passwords in this file has been used by the worm.

The use of the `.rhosts` file can also post a risk to security. This fact has been exploited by the worm.

And lastly, as a quotation before ending,

> Robert T. Moris was convicted of violating the computer Fraud and Abuse Act (Title 18), and sentenced to three years of probation, 400 hours of community service, a fine of $ 10,050, and the costs of his supervision. His appeal, filed in December, 1990, was rejected the following March.
>
> *Zen and the Art of the Internet*, by Brendan P. Kehoe

§ E.17 RLS algorithm for object-location problem

During the course of my study in the M.Sc. course Control and Information Technology at UMST in 1994, I did a design exercise on object-location problem with Dr. Martin Zarrop. This paper was written more than two years later while I was in Japan. I presented this at a conference in Singapore, and also a second time in an in-house seminar at the Furuta Laboratory, TIT, Tokyo. This latter presentation was given on 3^{rd} January 1998 between 4 and 5 pm.

Proceedings of the World Congress on Systems Simulation, September 1–3, 1997, Singapore

Simulation Techniques using RLS Algorithm for Object-location Problem

Kittisak Tiyapan†

Abstract

This work studied the application of recursive least square algorithm to object-location (OL) problem, that is the problem of locating an object in a noisy environment. By using RLS algorithm with random search and back-stepping techniques in simulations, some good results had been achieved.

Introduction

Performance function of a system can at times be difficult to identify. In practice it often suffices to keep a plant operating at the optimal performance without knowing what the actual performance function is.

In this work, control of plants with unknown performance functions subjected to noises are simulated using an extremum controller. The controller uses model-based hill-climbing technique and recursive least square algorithm. Theory, as well as applications of various control schemes, can be found in reference (Wellstead and Zarrop, 1991). Convergence properties, for example convergence with probability one, possible convergence points, and asymptotic behaviour of recursive algorithms have been made possible by relating these algorithms to deterministic differential equations (Ljung, 1977). One example is the analysis of convergence properties given in (Zarrop and Rommens, 1993) for a multi-input adaptive extremum controller.

A problem in object-location can be thought of as being one of finding the top of a badly shaped hill. In this work, a recursive least square (RLS) algorithm is used together with a random search and back-stepping. The reason in using an RLS algorithm is because the RLS algorithm is good when the measurement is subject to noises. Also, the convergence of the RLS algorithm is guaranteed when the performance function is holomorphic in $D \in R$.

Object-location problem

The problem is one of finding the location of an object within a given area when measurement is subject to noise. This problem can be considered as being that of finding the summit of a hill which looks like a single spike surrounded by flat area. The height of the hill is represented by the overlapping area between a frame or mask of a fixed size, which is built around the top of our quadratic model, and the object itself. (See Figure 1) The quadratic models used for x-coordinate and y-coordinate are

$$\mathrm{Mdl}_1 = a_1 u_1^2 + b_1 u_1 + c_1, \tag{1}$$

$$\mathrm{Mdl}_2 = a_2 u_2^2 + b_2 u_2 + c_1, \tag{2}$$

where u_1, u_2 are x- and y-coordinates respectively, $u_1, u_2 \in R$; $a_i, b_i, c_i \in R\; i = 1, 2$ are coefficients. From Equation 1 and 2 the summit is at

$$x = -\frac{b_1}{2a_1}, \tag{3}$$

$$y = -\frac{b_2}{2a_2}. \tag{4}$$

---

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Then, centred around this summit, a square mask with definite size, $\mathrm{Msk}(x, y)$, is created. In other word

$$\mathrm{Msk}(u_1, u_2) = \left\{(u_1, u_2) \in R^2 | x - d \leq u_1 \leq x + d, y - d \leq u_2 \leq y + d, d \in R\right\}.$$

Let $\mathrm{Ovp}(u_1, u_2)$ be the area of the intersection between a given object $\mathrm{Obj}(u_1, u_2)$ and the mask $\mathrm{Msk}(u_1, u_2)$ which indicates how close the estimation is to the object, that is

$$\mathrm{Ovp}(u_1, u_2) = \mathrm{Obj}(u_1, u_2) \bigcap \mathrm{Msk}(u_1, u_2) + \text{noise},$$

where the noise is random with normal distribution and having the mean $\mu = 0$.

Algorithm 1 is used in carrying out the simulation. Object and masks are considered as comprised of square pixels of equal size.

Results

Figure 2 shows the result when recursive least-square algorithm is applied together with a random search. The variance of the noise is $\sigma^2_{\text{noise}} = 0.1$, and the variance of the dither signal is $\sigma^2_{\text{dither}} = 0.02$. Here the result shows that the time required to recover the lost of object can be unacceptably long.

[Figure 2 used to be here.]

Figure 2 *The object (top left), graphical representation of the overlapping area (top right), x coordinate estimates, and y coordinate estimates.*

Figure 3 indicates that the result is greatly improved after a back-stepping routine is introduced. Figure 3 (a) has got $\sigma_n se^2 = 0.1$, $\sigma^2_{\text{dither}} = 0.3$, while Figure 3 (b) has got $\sigma^2_{\text{noise}} = 0.03$, $\sigma^2_{\text{dither}} = 0.01$. In Figure 3 (b) the result is very good, thus showing that the back-stepping technique is helpful. By comparing Figure 3 (b) with Figure 3 (a), it can be seen that both the variances of the noise and that of the dither signal has an influence upon the performance of the algorithm.

[Figure 3 used to be here.]

Figure 3 *After introducing back-stepping routine.*

Random search routines need to be introduced into the algorithm for the object-location problem because there is an area of flatness around the hill. Back-stepping routines makes possible tracking of a badly-shaped performance function as is the case here. There are other possibility that could help improving the performance of the algorithm, for example with the help of previous information one could vary the area for random search, hence faster algorithm.

Discussion

A possible practical problem that could be solved by the proposed algorithm could be that of finding the position of a high temperature object within a tank by detecting its heat wave. Suppose that the fluid within the tank and which surrounds the object is, on average, of lower temperature than the object. But there exists temperature fluctuation which is the result of convective movement of the fluid, and which results in thermal noise.

Conclusion

By combining the RLS algorithm with some suitable routines it is possible that the RLS algorithm can still be useful in more difficult situations where analyticity condition is not met and there is a large area where the slope of the performance function is zero.

Appendix

Algorithm 1. This is the algorithm used in the above section on object-location problem.

```
S0  $t = 0$;
    $\hat{a}_1(0)$, $\hat{a}_2(0)$, $\hat{b}_1(0)$, $\hat{b}_2(0)$, $\hat{c}_1(0)$, $\hat{c}_2(0)$, $\mathbf{P}_{6\times6}(0)$
S1  $u_1(t) = -\frac{\hat{b}_1}{2\hat{a}_1} + v(t)$
    $u_2(t) = -\frac{\hat{b}_2}{2\hat{a}_2} + v(t)$
S2  $(u_1, u_2)$ is mid-point of the mask
    $\mathrm{Ovp}(u_1, u_2)$ = overlapped pixels
S3  if $\mathrm{Ovp}(u_1, u_2) > 1$, then
        $t = t + 1$
        $\mathbf{P}(t) = \frac{1}{\lambda}\mathbf{P}(t-1)\left\{\mathbf{I} - \frac{\hat{\xi}(t)\hat{\xi}^T(t)\mathbf{P}(t-1)}{\lambda + \hat{\xi}^T(t)\mathbf{P}(t-1)\hat{\xi}(t)}\right\}$
        $\hat{\theta}(t) = \hat{\theta}(t-1) + \mathbf{P}(t)\hat{\xi}(t)\left\{y(t) - \hat{\xi}^T(t)\hat{\theta}(t-1)\right\}$
    elseif the number of steps consecutively stepped back not more than 3 consecutive steps,
      retrace another step
    else random search for $u_1$ and $u_2$
```

```
        $\hat{b}_1 = -2a_1u_1$
        $\hat{b}_2 = -2a_2u_2$
    S4  goto Step 1
```

In the above algorithm the recursive least-square algorithm is used together with random searching and back-stepping.

Here $v(t)$ is dither signal which is zero-mean random signal with normal distribution, $\hat{\xi}(t) = [u_1(t), u_2(t)]^T$ where $u_1(t)$, $u_2(t)$ are x- and y-axis at time step t respectively,

$$\hat{\xi}(t) = \left[u_1^2(t-1), u_2^2(t-1), u_1(t-1), u_2(t-1), 1, 1\right]^T$$

$$\hat{\theta}(t) = \left[\hat{a}_1(t), \hat{a}_2(t), \hat{b}_1(t), \hat{b}_2(t), \hat{c}_1(t), \hat{c}_2(t)\right]^T$$

Algorithm 1 can be visualised by the diagram in Figure 4.

[Figure 4 used to be here.]

Figure 4 *Flow-chart of the algorithm described above.*

Acknowledgement

The author thanks Dr. M. B. Zarrop at the Control Systems Centre, Universith of Manchester Institute of Science and TEchnology, for many of his good suggestions for this project.

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§ E.18 Modelling the economics

Modelling the Economics

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Abstract

Abrupt or rapid changes happened often through out the history of economics. Despise the fact that they are very important they are still not well understood. Attemps to simplify them down to mathematical equations have proved to be difficult. This paper suggests that simulation on a random networks model should be a more appropriate approach.

Introduction

Early study in economics deals with only one or few things at a time. For example one studies about population, ratio of land to population or the trend in GNP of a nation. This proved to be both ineffective as well as misleading. Later studies in economics emphasise more and more on the interrelation among several factors (Forrester, 1971). In this approach the economics is studied as being a complex multivariable systems. Though the analysis from such approach is much more satisfactory it is still not possible to predict the future of the economics of a nation. There are even situations which can not be explained. An example of these is hyperinflation. Among the most baffling occurences of hyperinflation in history are that which happened to Hungary after both world wars and that which happened to China after the second world war. A more recent example of hyperinflation is that which occured in eastern european countries for example, the inflation in Poland which rose above 600 percents in 1989 (Fisher *et al*, 1996). While such abrupt transitions in economics have been puzzling economists for centuries, physicists have been more accustomed to them. Scientists in the field of physics and chemistry have more than one way to explain a phase-transition. Such a phase transition is not a result of any single decision of the government but, rather it is a result of various accumulated potentials. Among the existing theorems which use a similar idea of effect of gathering potentials is the *cobweb* theorem (Davis, 1980). A model which can explain the phase-transition situations in economics will greatly increase our understanding in economics. In this work it has been suggested that a tool which has proved itself to be useful in physics can also be applied to the study of economics.

Theory

Economics is a complex systems. It is a system because it is governed by certain rules. It is complex because there are so many factors involved and the rules are complicated. Systems which has got such complicated nature and covers a great many individual components can be best represented as a geometrical networks of components and interactions among them.

Figure 1 and Figure 2 shows the stereotypical features found in various fields of science. Here the capacity is the capacity of the underlying networks. The probability is the likeliness that each component within the networks be in either one of the two opposite states.

In a very large network the phase change is abrupt, the rate of change close to the critical point approaches infinity as shown in Figure 1. Such situation exists only in theory. In the study of economics only Figure 2 is applicable since the amount of resources are limitted thus rendering an infinite growth in either way impossible.

Figure 2 shows two distinct features which are the exponentially increasing part and the part where saturation occurs. Both of these features are those normally found in economics.

Proposition 2.1. *Quantitive factors in economics can always be eplained by using a phase-transition model as the one shown in Figure 2.*

Proposition 2.2. *Qualitative factors which are the potentials and the driving forces behind economic changes comprise other things as well as education, unemployment, forestry condition, environmental condition, militarianity, epidemics, spending habits of population, extense and level of corruption in government or management, morality, occurrences of crimes, drug problem, productivity and war.*

Proposition 2.3. *Every abrupt change in economics comes from gathering potential of all the quantitative and qualitative factors involved.*

Proposition 2.1, 2.2 and 2.3 mention about two kinds of factors in economics namely, those which are qualitative and those which are quantitative in nature. In order to be able to make a computer model it is necessary to quantitise qualitative factors. Also the structure of the network needs to be identified. These are among those tasks to be carried out in the following stage of this study.

Conclusion

Any model which can explain extreme situations in economics is a better model than existing ones. Such a model will enable a better prediction of economical future via simulation as well as allow better understanding in the subject.

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§ E.19 Money flow in networks

Modelling of economics as a flow of money within networks
Kittisak Tiyapan†
Short title: Modelling economics networks
4^{th} November 1997

Abstract

The study of economics comprises of two main branches namely, macroeconomics and microeconomics. The model used for simulation in this work take into account parameters from both branches at the same time. Therefore it is a good representation of the real systems. The model is centred around people instead of centred around money as most of the conventional mathematical models do. This would reduce unwanted effects which are related to the latter approach for example, *money illusion.* Such proposed model can be used for simulation. The results of the simulations may then be used for an analysis purpose or for decision-making.

Introduction

Most of the existing economics models are good for situations close to economic equilibrium or situations where any changes are gradual. But changes in economic situations are sometimes abrupt. These fast phenomena are important and cannot be neglected. They reflect the results of accumulated potential from various sources affecting the economics. An example of such fast phenomena is hyper-inflation. In Hungary in July 1946, for example, inflation rose as high as 4×10^{16}.

The *percolation* theory is useful in representing many things in nature. It finds applications in Physics (Sen, 1997; Bershadskii, 1997), Chemistry (Bychkov *et al*, 1996; Tremblay *et al*, 1996; Shih and Reiser,

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1997), Cosmology (Zeldovich, 1983; Kazakov, 1989; Seiden and Schulman, 1990), Biology (Wu and Bradley, 1991; Gerardi and Romiti, 1991; Calzolati *et al*, 1991), *etc.* Characteristic to the theory is an abrupt jump at the onset of percolation which is caused by gathering potentials in the background. The location of the point of percolation depends on the structure of the network considered. Figure 1 shows the stereotypical characteristic mentioned. The saturation part in Figure 1 (b) is there because of the limitation of resources or to be more precise, the limitted size of the network. In a network of indefinite size the jump goes to infinity.

Examples of these two phases are the two states of currency, namely hard currency and non-hard currency.

Game-theory is useful in modelling economics when the number of players is small. In game theory players interact among one another. In a normal situation in society two people could live far apart from each other and never interact in anyway. In fact this is usually the case. By representing a society by a network having people as the nodes game-theory can still be useful in describing any area covering nodes which are neighbours to one another. But though the results from extensive analysis on small scales using game-theory can play a part in the analysis of the overall network, game-theory is not a necessary part in the study of networks.

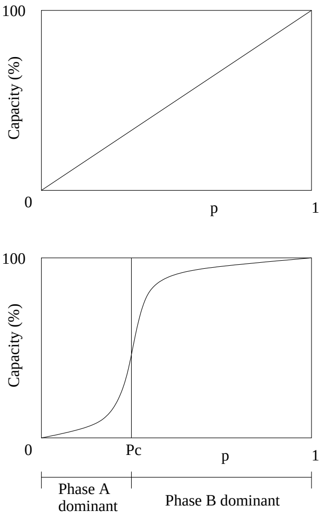

Figure 1. Potential gathers linearly in (a), but the sign only shows as a characteristic jump *(b)* at the point dividing dominances of the two phases.

Theory

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Theory

Definition 1. *Consider* Nt^n*, a network of* n *dimension. Let* Ct_i^m*,* $i = 1, 2, \ldots, q$ *be the* i^{th} *cluster of m-dimension among a total of* q *clusters each having a specific characteristic of interest. Then the critical probability of the overall network is expressed as*

$$\left\{p_c: \quad \exists j \in N \quad s.t.\ Ct_j^m \bigcap Sf_{\inf x}^{n-1} \neq \emptyset \quad and Ct_j^m \bigcap Sf_{\sup x}^{n-1} \neq \emptyset\right\} \tag{1}$$

According to Definition 1 at critical point of a 2-dimensional network there is a single cluster which spans the network from the left-extreme to the right-extreme line or, from the top-extreme to the bottom-extreme line. Similarly in 3-dimensional networks the cluster spans the whole length of either one of the three coordinate axes.

Definition 2. *A society is a network of n dimensions represented by $Sc^n(Pl, Mf)$ where, people (Pl) are nodes and paths in which money flows (Mf) are bonds of the network. The n dimensions are n ways in which people are positioned for example, social status, amount of income or iducational level.*

This means that a society is represented by a network which is comparable to a network of pipes. A junction between pipes is a person. The fluid flowing through the pipes is essentially money though, it may include other media of transactions which have lower liquidity as well.

According to Definition 2 a society is a network with time-variant structure. When a person changes his job, for example, he is moved into another position. Also people born and die, therefore the amount of nodes as well as their positions are changed.

Due to the time-invariant nature mentioned above it is useful to use sampling-and-hold technique when studying the economics networks.

Definition 3. *The network for each time-step k is*

$$\left\{ Sc_k : \quad Sc_k = Sc\left(Pl(k-1), Mf\left(k-1, \bigcap_{k-1<t<k} j(t) \right) \right), \quad k = 1, 2, \ldots \right\} \tag{2}$$

Definition 4. *The coordination number of each node is*

$$\{Co(i) : \quad Co(i) = n, \quad \forall Pl(i), \quad \exists Mf(i,j), \quad \forall j = 1, \ldots, n, \quad \nexists Mf(i,j) \quad s.t. \quad j > n\} \tag{3}$$

The coordination number according to Definition 4 can have a very high value.

Proposition 1. *The economical well-being of a society depends on many factors (potentials) which comprises among other things, education, investment, employment, environmental soundness. These factors in turn are reflected in the merit quality of individuals for example, moral, morale, efficiency, attitude and lifestyle. And the coordination number is also related to all the merit factors mentioned. Therefore the economical well-being of a society is reflected in the value of the coordination number.*

Proposition 2. *One can control these potentials but, one can not control the actual jump. In order to control the jump one could do the following.*

- *Control the potentials before the jump occurs.*
- *Build the potentials in the opposite direction.*
- *Enlarge the network.*

Enlarging the network can be done by increasing the resources involved.

Proposition 3. *There is an optimal value for the coordination number for all economical networks.*

Methodology

In Figure 2 certain assumptions must be made when transferring parameters of the real world into those of the model. Here the model is rather a computer model than a mathematical one.

People in the real world are mapped into components within the networks in the distributed random fashion. Groups of people are then represented by groups of components, namely clusters. Examples are companies which could merge to form a larger company much the same way as clusters do to form a larger cluster. Neither component represents any specific person in the real world, nor cluster any specific company. Yet the model as a whole represents the real world, as is reflected in the same statistical values which both of them shares.

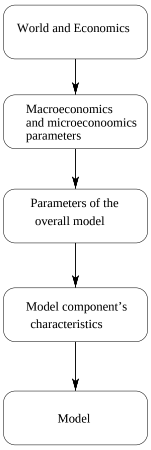
World and Economics
Macroeconomics and microeconoomics parameters
Parameters of the overall model
Model component's characteristics
Model

Figure 2. Relationship between the world and the model.

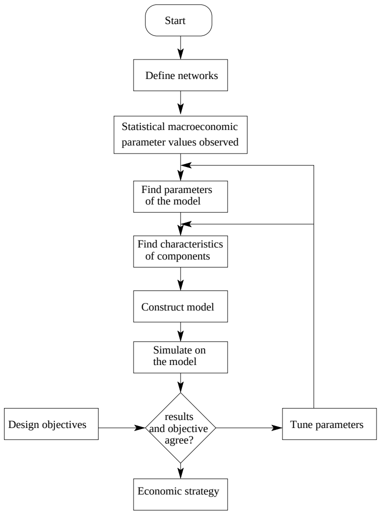
Start
Define networks
Statistical macroeconomic parameter values observed
Find parameters of the model
Find characteristics of components
Construct model
Simulate on the model
Design objectives
results and objective agree?
Tune parameters
Economic strategy

Figure 3 shows both the procedure for carrying out a simulation.

Conclusion

Definition of money does change. But when people nowaday starve they do so the same way people two thousand years ago did. Therefore a model of the world which is centred around people is better than those centring around money. Further work is to do simulations based on the idea presented and some recorded history and then compare the results with what actually occured.

Figure 3. Steps for economics decision making via simulations on a model.

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§ E.20 Modelling of traffic congestion

Modelling of traffic congestion

Kittisak Tiyapan†

Short title: Modelling of traffic congestion
4^{th} November 1997

Abstract.

Bad traffic conditions could have adverse effects on a country's economics and well-beings. News of such devastating situations as a traffical standstill means reduced competitive edges for a developping country, discouraged tourists and an enormous amount of wasted resources. Percolation theory has long been associated with similar problems occurring in other areas of science, for example sieve-blinding and filtration. These problems essentially deals with the flow of fluids through a media which has random structure and, when the media is subjected to blockages. Similar to how a filter media can be classified according to the way it behaves at the onset of percolation, any traffical networks could be distinguished from the others by its ability to withstand heavy traffic conditions before a standstill occurs. This work is the beginning of an attempt to model traffical networks with the idea of percolation in mind. It has been proposed here that a traffical networks can either be excellent at tolerating a standstill or, it can boast that it rarely finds itself congested, but never both at the same time. It has been proposed that a trade-off has to be made if one were to design a new traffical networks.

Introduction

there are two kinds of lattices or tessellations. One is regular and the other random. Examples of regular tessellations are honeycomb and Kagomé lattices in 2-dimension and hexagonal tessellations in 3-dimension. Tessellations which are not regular are random (Chan and Ng, 1988; Meer and Connelly, 1989; Heinrich and Schule, 1995). There are random tessellations with underlying rules as a Voronoï tessellations (Boots, 1987; Muche, 1996; Riedinger *et al*, 1988). Examples of these rule-specific random tessellations are patterns of cells in living tissue and structure of the universe (van de Weygaert, 1994). An example of these purely random tessellations is a traffical networks.

Critical probability (Jensen and Guttmann, 1996; Chayes, 1996; Jensen and Guttmann, 1995) is a property which has long been associated with the study of tessellations' properties. This value is characteristic for each type of tessellations concerned and is closely related to the coordination number of the tessellations or, in the random tessellations case, the average coordination number. For all kind of tessellations finding the critical probability can be done by doing simulations. Theoretical computation (Shante and Kirkpatrick, 1971) of exact values of critical probabilities is also possible but, only for certain types of regular tessellations. So for traffic networks the only two possible ways of finding the critical probability for a network of a certain city are by doing simulations on computer and, by approximating the value from the average coordination number of the network.

Topics related to traffic congestions and the states of congestion have been discussed in existing literatures (Sherali *et al*, 1997; Sisiopiku, 1997; Hennessy and Wiesenthal, 1997). In this work it has been proposed for a traffical networks of a specific city that, once the number of roads experiencing a stoppage reaches a certain value the whole city will come to a standstill. It has also been further proposed that, in the designing of a traffical networks a trade-off has to be made between robustness to congestions and robustness to standstills. This means that we can have a traffic networks which is most of the time congested but never comes to an overall standstill or, we can have a traffic networks which is most of the time flowing freely until the amount of cars increases causing it to come to a overall standstill, but we can not have a kind of traffic networks which free-flowing all the time and never experience a standstill.

Study of traffic networks based on queueing theory also categorise traffic into several states (Leutzbach, 1988) but, the definitions of these states are quite different from those here.

Theory

This section is organised as a number of definitions leading to main theorems. These definitions are good because they serves as definitive description to the problem as well as aid for writing a program for doing simulation.

Definition 1. *A network is the linkages of vertices by bonds and has bond percolation probability and vertex percolation probability as its two invariances. A network will be denoted by*

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$\mathrm{Nwk}(\mathrm{Vtx},\mathrm{Bnd},P_{\mathrm{Vtx}},P_{\mathrm{Bnd}})$ *where* $\mathrm{Vtx},\mathrm{Bnd} \subset N + \{+\infty\}$, $P_{\mathrm{Vtx}}, P_{\mathrm{Bnd}} \in R$, $0 < P_{\mathrm{Vtx}} < 1$ *and* $0 < P_{\mathrm{Bnd}} < 1$.

Definition 2. $\mathrm{Nwk}(\mathrm{Bnd})$ *represents a network which is considered as connections of bonds at vertices while* $\mathrm{Nwk}(\mathrm{Vtx})$ *represents a network which is considered as connections of vertices by bonds.*

Definition 3. $\mathrm{Bnd}(V)$ *represents a set of all the bonds which are connected to* v, $\forall v \in V$, $V \subset \mathrm{Vtx}$. *Similarly* $\mathrm{Vtx}(B)$ *represents a set of all the vertices which are connected to* b, $\forall b \in B$, $B \subset \mathrm{Bnd}$.

Definition 4. *A bond can have either of the two states which are called free,* Bnd^0, *or blocked,* Bnd^1. *Likewise a vertex can be either free,* Vtx^0, *or occupied,* Vtx^1.

Definition 5. *A cluster is a set of either of the two components of a network, namely vertices and bonds, with each one of its members connecting to some other members. There are clusters of bonds represented by*

$$\{\mathrm{Cl}(\mathrm{Bnd}) : \forall b_1 \in \mathrm{Cl}(\mathrm{Bnd}), \exists b_2 \in \mathrm{Cl}(\mathrm{Bnd}) \ s.t.\ \mathrm{Vtx}(b_1) = \mathrm{Vtx}(b_2), \mathrm{Cl}(\mathrm{Bnd}) \subset \mathrm{Bnd}\} \tag{1}$$

and there are clusters of vertices represented by

$$\{\mathrm{Cl}(\mathrm{Vtx}) : \forall v_1 \in \mathrm{Cl}(\mathrm{Vtx}), \exists v_2 \in \mathrm{Cl}(\mathrm{Vtx}) \ s.t.\ \mathrm{Bnd}(v_1) = \mathrm{Bnd}(v_2), \mathrm{Cl}(\mathrm{Vtx}) \subset \mathrm{Vtx}\}. \tag{2}$$

Definition 6. *The set of all the clusters (of either bonds or vertices) is*

$$\{\mathrm{Tcl}(\cdot) : \quad \mathrm{Cl}(\cdot) \subset \mathrm{Nwk}(\cdot) \Longleftrightarrow \mathrm{Cl}(\cdot) \in \mathrm{Tcl}(\cdot)\}, \tag{3}$$

where $\cdot$ *is either* Bnd *or* Vtx.

Definition 7. *Each cluster has a definite size which is the number of either bonds or vertices comprising it, depending on whether the cluster is a cluster of bonds or a cluster of vertices. This size is represented by*

$$\mathrm{Size}(\mathrm{Cl}) \quad \mathrm{Cl} \in N + +\infty. \tag{4}$$

Definition 8. *The width of a cluster is the dimension of the cluster in either* x- *or* y-*axis and, is represented by*

$$\mathrm{Width}(\mathrm{Cl}) \in R \quad \mathrm{Cl} \in N + +\infty. \tag{5}$$

Definition 9. *The largest cluster (of either bonds or vertices) of any network is*

$$\{\mathrm{Mcl}(\cdot) : \quad \mathrm{Size}(\mathrm{Mcl}) = \sup(\mathrm{Size}(\mathrm{Cl})) \quad \forall\, \mathrm{Cl} \in \mathrm{Tcl}\}. \tag{6}$$

Definition 10. *The percolating cluster of any network is*

$$\{\mathrm{Pclust}(\cdot) : \quad \mathrm{Pclust}(\cdot) = \mathrm{Mcl} \quad \mathrm{Mcl}(\cdot) \longrightarrow \infty \quad as \quad \mathrm{Size}(\mathrm{Nwk}(\cdot)) \longrightarrow \infty\}. \tag{7}$$

While the smallest percolating cluster possible for any network is

$$\mathrm{Inf}(\cdot) = \inf(\mathrm{Pclust}). \tag{8}$$

Definition 11. *The percolating cluster of a traffic network is*

$$\{\mathrm{Pcl}(\mathrm{Bnd}) : \quad \mathrm{Pcl}(\mathrm{Bnd}) = \mathrm{Cl}(\mathrm{Bnd}) \quad \mathrm{Width}(\mathrm{Cl}(\mathrm{Bnd})) = \mathrm{Width}(\mathrm{Nwk}(\mathrm{Bnd}))\}. \tag{9}$$

Definition 12. *For any network a crititcal probability is*

$$\left\{\mathrm{Pc}(\cdot) : \quad \mathrm{Pc}(\cdot) = \frac{\sum_{\forall\,\mathrm{Cl}\in\mathrm{Tcl}} \mathrm{Size}}{\mathrm{Size}(\mathrm{Nwk}(\cdot))} \quad \mathrm{Mcl}(\cdot) = \inf(\cdot)\right\}. \tag{10}$$

While $\mathrm{Pc}(\mathrm{Bnd})$ *is its bond critical probability and* $\mathrm{Pc}(\mathrm{Vtx})$ *is its vertex critical probability.*

Definition 13. *For a traffical network a critical probability is*

$$\left\{\mathrm{Pc}(\mathrm{Bnd}) : \quad \mathrm{Pc}(\mathrm{Bnd}) = \frac{\sum_{\forall\,\mathrm{Cl}\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}))}{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}))} \quad \mathrm{Mcl}(\mathrm{Bnd}) = \mathrm{Pcl}(\mathrm{Bnd})\right\}. \tag{11}$$

Definition 14. *A free-flowing network is*

$$\{\mathrm{Nwk} : \quad \exists \mathrm{Pcl}(\mathrm{Nwk}(\mathrm{Bnd}^0)) \quad \nexists \mathrm{Pcl}(\mathrm{Nwk}(\mathrm{Bnd}^1))\}. \tag{12}$$

A stand-still is

$$\{\mathrm{Nwk} : \quad \nexists \mathrm{Pcl}(\mathrm{Nwk}(\mathrm{Bnd}^0)) \quad \exists \mathrm{Pcl}(\mathrm{Nwk}(\mathrm{Bnd}^1))\}. \tag{13}$$

A congested network is

$$\{\mathrm{Nwk}: \quad \nexists \mathrm{Pcl}(\mathrm{Nwk}(\mathrm{Bnd}^0)) \quad \nexists \mathrm{Pcl}(\mathrm{Nwk}(\mathrm{Bnd}^1))\} \oplus \{\mathrm{Nwk}: \quad \exists \mathrm{Pcl}(\mathrm{Nwk}(\mathrm{Bnd}^0)) \quad \exists \mathrm{Pcl}(\mathrm{Nwk}(\mathrm{Bnd}^1))\}\,. \tag{14}$$

The following definitions concern with features found in typical traffic networks.

Definition 15. *A* road *is a single bond. It can be either a one-way or a two-way street but, without any branching along its length. The only two opennings are located at both ends.*

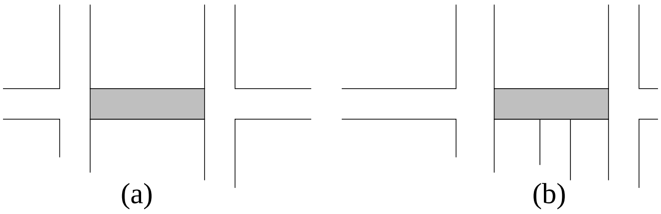

Figure 1 *The dark line shown in (a) is a road while that shown in (b) is not.*

Definition 16. *At any instance a road is said to be* congested *when no cars could enter it.*

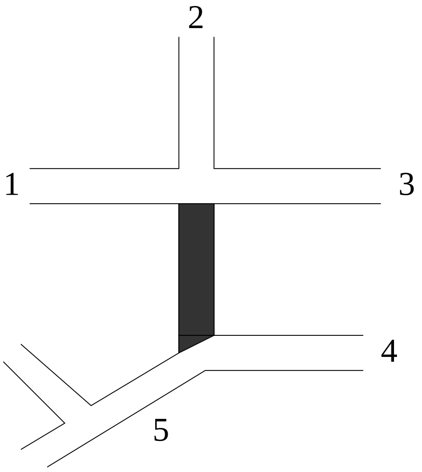

Figure 2 *The road portrayed is congested whenever no cars could enter it from any of the five adjacent roads.*

Definition 17. *A round-about is considered simply as being a vertex.*

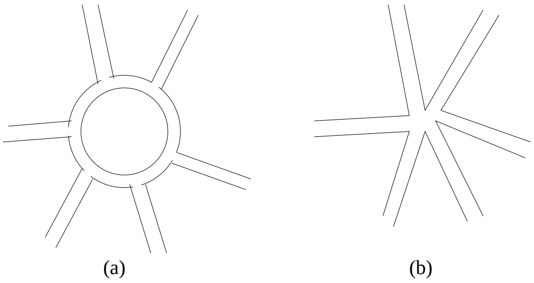

Figure 3 (a) *is considered to be the same as* (b).

Definition 18. *Let* $p = \frac{\sum_{\forall\,\mathrm{Cl}\in\mathrm{Tcl}}\mathrm{Size}(\mathrm{Cl}(\cdot))}{\mathrm{Size}(\mathrm{Nwk}(\cdot))}$ *then,*

$$\mathrm{Frf}(\mathrm{Nwk}(\mathrm{Bnd})) = \{\mathrm{Nwk}(\mathrm{Bnd}): \quad p \in [0, \min(\mathrm{Pc}, 1-\mathrm{Pc})]\}, \tag{15}$$

$$\mathrm{cgd}(\mathrm{Nwk}(\mathrm{Bnd})) = \{\mathrm{Nwk}(\mathrm{Bnd}): \quad p \in [\min(\mathrm{Pc}, 1-\mathrm{Pc}), \max(\mathrm{Pc}, 1-\mathrm{Pc})]\}, \tag{16}$$

and

$$\mathrm{Stp}(\mathrm{Nwk}(\mathrm{Bnd})) = \{\mathrm{Nwk}(\mathrm{Bnd}): \quad p \in (\max(\mathrm{Pc}, 1-\mathrm{Pc}))\}, \tag{17}$$

Theorem 1. *For a traffical network*

$$\mathrm{Pc}(\mathrm{Bnd}^0) = \mathrm{Pc}(\mathrm{Bnd}^1), \tag{18}$$

where Bnd^0 *is a free bond or a non-congested road and,* Bnd^1 *is an occupied bond or a congested road.*

Proof.: The networks ramains the same whether one consider it as a network of free roads or, that of congested roads. In other words,

$$\mathrm{Nwk}(\mathrm{Bnd}^0) = \mathrm{Nwk}(\mathrm{Bnd}^1) = \mathrm{Nwk}(\mathrm{Bnd}) \tag{19}$$

When $\mathrm{Mcl}(\mathrm{Bnd}^0) = \mathrm{Pcl}(\mathrm{Bnd}^0)$,

$$\mathrm{Pc}(\mathrm{Bnd}^0) = \frac{\sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^0)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^0))}{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}^1))}. \tag{20}$$

While when $\mathrm{Mcl}(\mathrm{Bnd}^1) = \mathrm{Pcl}(\mathrm{Bnd}^1)$,

$$\mathrm{Pc}(\mathrm{Bnd}^1) = \frac{\sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^1)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^1))}{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}^0))}. \tag{21}$$

Now suppose instead of starting from $\mathrm{Nwk}(\mathrm{Bnd}^0)$ and keep adding Bnd^1 randomly until $\mathrm{Mcl}(\mathrm{Bnd}^1) = \mathrm{Pcl}(\mathrm{Bnd}^1)$, we could have said from the start that we were looking at $\mathrm{Nwk}(\mathrm{Bnd}^1)$ and added Bnd^0 randomly until $\mathrm{Mcl}(\mathrm{Bnd}^0) = \mathrm{Pcl}(\mathrm{Bnd}^0)$. From this reasoning

$$\mathrm{Mcl}(\mathrm{Bnd}^1) = \mathrm{Pcl}(\mathrm{Bnd}^1) = \mathrm{Pcl}(\mathrm{Bnd}^0) = \mathrm{Mcl}(\mathrm{Bnd}^0). \tag{22}$$

From Equation 19 we have

$$\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}^0)) = \mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}^1)) = \mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd})). \tag{23}$$

Thus it follows from Equation 20 and Equation 21 that

$$\mathrm{Pc}(\mathrm{Bnd}^0) = \mathrm{Pc}(\mathrm{Bnd}^1). \tag{24}$$

q.e.d.

q.e.d.

Theorem 2. *For a traffical network, when* $\mathrm{Mcl}(\mathrm{Bnd}^1) = \mathrm{Pcl}(\mathrm{Bnd}^1)$

$$\mathrm{Pc}(\mathrm{Bnd}^0) = P_{\mathrm{Bnd}^0} = 1 - \frac{\sum_{\forall\,\mathrm{Cl}\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^0))}{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}))}. \tag{25}$$

Proof.: When $\mathrm{Mcl}(\mathrm{Bnd}^1) = \mathrm{Pcl}(\mathrm{Bnd}^1)$ we know from Equation 21 that

$$\mathrm{Pc}(\mathrm{Bnd}^1) = \frac{\sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^1)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^1))}{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}^0))}. \tag{26}$$

Now the only states possible for a Bnd are either Bnd(0) or Bnd(1). Thus

$$\sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^0)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^0)) + \sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^1)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^1)) = \mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd})) \tag{27}$$

or,

$$\sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^0)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^0)) = \mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd})) - \sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^1)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^1)). \tag{28}$$

Then

$$\begin{aligned}
\mathrm{Pc}(\mathrm{Bnd}^0) = P_{\mathrm{Bnd}^0} &= \frac{\sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^0)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^0))}{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}))} && (29)\\
&= \frac{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd})) - \sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^1)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^1))}{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}))} && (30)\\
&= 1 - \frac{\sum_{\forall\,\mathrm{Cl}(\mathrm{Bnd}^1)\in\mathrm{Tcl}} \mathrm{Size}(\mathrm{Cl}(\mathrm{Bnd}^1))}{\mathrm{Size}(\mathrm{Nwk}(\mathrm{Bnd}))} && (31)
\end{aligned}$$

q.e.d.

Theorem 3. $\forall\,\mathrm{Nwk(Bnd)}$, Frf(Nwk(Bnd)) *is a free-flowing network,* cgd(Nwk(Bnd)) *is a congested network and* Stp(Nwk(Bnd)) *is a standstill.*

Proof.: Consider a traffical network with $\mathrm{Pc} < 0.5$ first as $\mathrm{Pc(Bnd^1)}$ in $\mathrm{Nwk(Bnd^0)}$ and then, as $\mathrm{Pc(Bnd^0)}$ in $\mathrm{Nwk(Bnd^1)}$. This is shown in Figure 4.

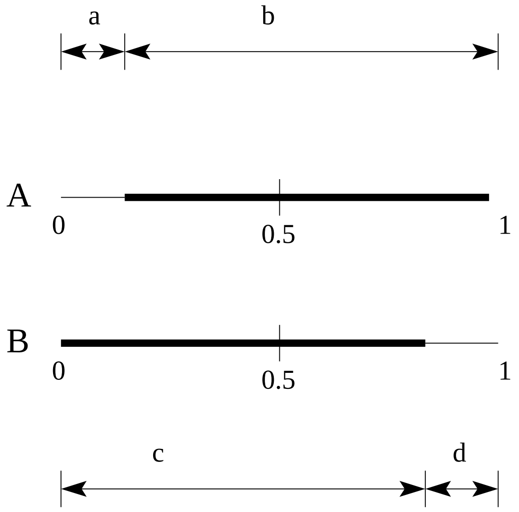

Figure 4. The case where $\mathrm{Pc} < 0.5$. Here a is the situation where cars have not yet percolated; b is that where cars have percolated; c is that where space has percolated and d is that where space has not yet percolated.

From Theorem 1 and Theorem 2 we have $a = d$ and $b = c$ in Figure 4. Then from Definition 18,

$$\mathrm{Frf(Nwk(Bnd))} = \{\mathrm{Nwk(Bnd)} : \quad p \in [0, a)\}\,, \tag{32}$$

$$\mathrm{cgd(Nwk(Bnd))} = \{\mathrm{Nwk(Bnd)} : \quad p \in [a, 1-a]\} \tag{33}$$

and

$$\mathrm{Stp(Nwk(Bnd))} = \{\mathrm{Nwk(Bnd)} : \quad p \in (1-a, 1]\}\,. \tag{34}$$

Then from Figure 4 Frf(Nwk(Bnd)) represents the situation where cars has not yet percolated while space has. From Definition 14 this represents a free-flowing network. Similarly cgd(Nwk(Bnd)) represents that situation where both cars and space have percolated and, Stp(Nwk(Bnd)) that situation where only cars have percolated. From Definition 14, cgd(Nwk(Bnd)) is a congested network and Stp(Nwk(Bnd)) a standstill. Now proceed on in the same line as above for the case where $\mathrm{Pc} = 0.5$ and that where $\mathrm{Pc} > 0.5$ which are shown diagrammatically respectively as Figure 5 and Figure 6.

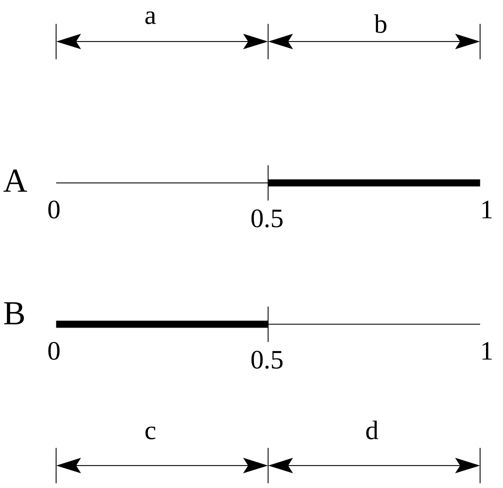

Figure 5. *The case where* Pc $= 0.5$.

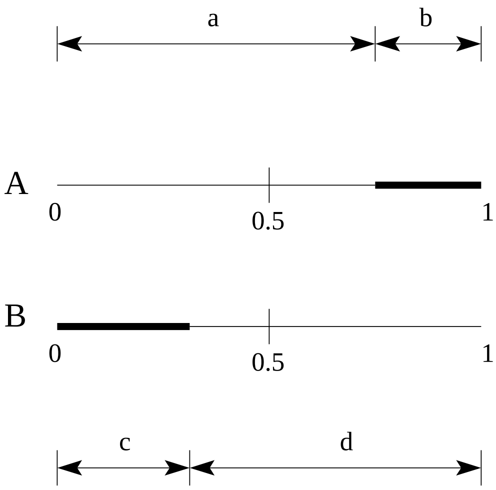

Figure 6. *The case where* Pc > 0.5.

In the case represented by Figure 5 Definition 18 gives

$$\mathrm{Frf}(\mathrm{Nwk}(\mathrm{Bnd})) = \{\mathrm{Nwk}(\mathrm{Bnd}) : \quad p \in [0, 0.5)\}, \tag{35}$$

$$\mathrm{cgd}(\mathrm{Nwk}(\mathrm{Bnd})) = \{\mathrm{Nwk}(\mathrm{Bnd}) : \quad p \in [0, 0.5] \quad \text{or} \quad \mathrm{Pc} = 0.5\} \tag{36}$$

and

$$\mathrm{Stp}(\mathrm{Nwk}(\mathrm{Bnd})) = \{\mathrm{Nwk}(\mathrm{Bnd}) : \quad p \in (0.5, 1]\}. \tag{37}$$

And Definition 14 says that Frf(Nwk(Bnd)) is a free-flowing network, cgd(Nwk(Bnd)) is a congested network and, Stp(Nwk(Bnd)) is a standstill. The juxtaposition between the congestion of space and cars is shown in [Figure] 7. And in the last case of Figure 6, Definition 18 again gives Equation 32, 33, and 34 while, Definition 14 again says that Frf(Nwk(Bnd)) is a free-flowing network, cgd(Nwk(Bnd)) is a congested network and, Stp(Nwk(Bnd)) is a standstill. The congestion of space and cars can then be shown in [Figure] 8.

Interval	**Car percolation**	**Space percolation**
Frf(·)	N	Y
cgd(·)	Y	Y
Stp(·)	Y	N

Figure 7 *The case where* Pc $= 0.5$.

Interval	**Car percolation**	**Space percolation**
Frf(·)	N	Y
cgd(·)	N	N
Stp(·)	Y	N

Figure 8 *The case where* Pc > 0.5.

q.e.d.

Algorithm

The algorithm for finding the critical probability for a bond network is shown in the diagram in Figure 9.

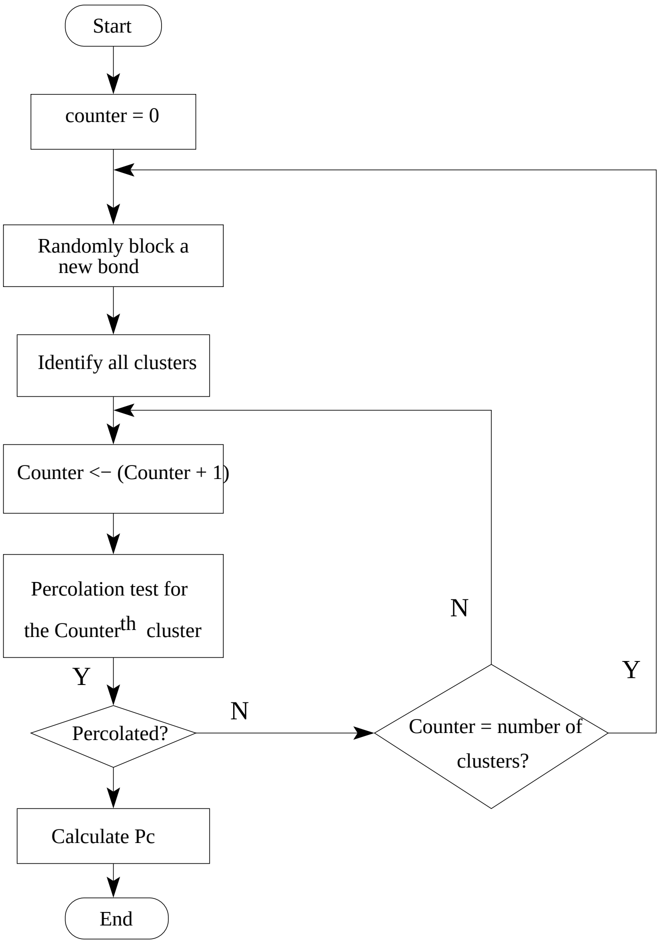
Start
counter = 0
Randomly block a new bond
Identify all clusters
Counter <− (Counter + 1)
Percolation test for the Counterth cluster
Y
Percolated?
N
Counter = number of clusters?
N
Y
Calculate Pc
End

Figure 9 *Algorithm for finding a bond percolation.*

Examples

This section shows the results of simulation on traffic networks. Both Figure 10 and Figure 11 are the traffic networks of Bangkok but larger area is covered by Figure 10.

[Figure 10 used to be here]

Figure 10 *[The percolating cluster. Bangkok from a map by Auto Guide Co. Pc = 0.587] Bangkok's traffic percolation simulation.*

[Figure 11 also used to be here.]

Figure 11 *[Pc = 0.678] Bangkok's innercity traffic percolation simulation.*

The critical probability obtained from simulations similar to these could be used to determine the states of traffic congestion of that city. Once the state of the traffic at any moment is known it will be possible [to] control the traffic accordingly. For example, once the number of cars in any area has nearly reached the percolation threshold it would be wise to stop traffic going into that area until the density has somewhat been reduced. Another example is that when there are two or more neighbouring areas experiencing heavy congestion it would be good to manage the traffic in such a way that moves cars away from those streets between those area as well as away from those congested area. This is to prevent the congested areas from merging [together] which could cause percolation.

Conclusion

Like the formation of galaxies when primordial gas has density exceeding certain value, and many things analogous to this which are found in plenty in nature, traffic in any city will come to a standstill once the number of congested roads reaches percolating value which is specific for each city.

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§ E.21 Critical probability in traffic modelling and control

Critical probability in traffic modelling and control

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2^{nd} April 1998

Abstract

Percolation has been playing a big role in many fields of Physics, Chemistry as well as Forestry and Biology. But up to this time its application to traffic modelling and control has not been mentioned in existing literature, contrary to its inherently similarity in nature to the field. This work is an attempt to introduce it to this application.

Introduction

Percolation says that for any kind of networks of connections among components, namely vertices and bonds, there is one parameter with its own unique value which could be different from other or other kind of networks. This parameter is named the *percolation probability* or the *critical probability.* This critical probability or p_c marks the transition from one of the two phases into another.

Theory

In order to be succinct, only one theorem will be briefly mentioned.

Theorem 1.For any traffic networks the state of traffic at any moment can be described as being in one of the three states, namely free-flowing, congested or stand-still. *Furthermore, the free- flowing case corresponds to the case where* $p < \min(1-p_c)$, *the congested case to the case where* $\min((1-p_c),p_c) \le p \le \max((1-p_c),p_c)$ *and the stand-still case to* $p \ge \max((1-p_c),p_c)$.

Proof.: From percolation theory we know that there exists $0 < p_c < 1$ for any networks. Obviously p_c can only have a value within one of these three intervals, that is < 0.5, $= 0$, or > 0. Furthermore, we may consider the percolation as being percolation of flowing roads as well as the percolation of jammed roads. Our interval from 0 to 1 can then be divided into three sections, namely $0 < p < \min(1-p_c)$, $\min((1-p_c),p_c) \le p \le \max((1-p_c),p_c)$ and $\max((1-p_c),p_c) < p < 0$. The first one of these three intervals corresponds with the situation where cars have not yet percolated while space has. And thus it corresponds to the free-flowing traffic. The next one corresponds either to the situation where both cars and space has percolated, or to the situation where neither has percolated, depending on whether $p_c < 0.5$ or $p_c > 0.5$ originally. And thus it corresponds to the congested traffic. And the last one corresponds to the situation where cars have percolated while space has not and therefore to the stand-still traffic. The case where $p_c = 0.5$ is a special case where the second one of the three intervals considered vanishes. □

Traffic Modelling

For modelling purpose, the percolation probability of the networks being considered is obtained by using the procedure shown in Figure 1.

The structures of the data mentioned in Figure 1 are given below.

Data structure 1. *Junction data in file named Junction. 3 datasets per record.* J*, the number assigned to this junction, of type* I^+*;* x*, x-coordinate of this junction, of type* R^+*;* y*, y-coordinate of this junction, of type* R^+*.*

Data structure 2. *Road data in file named Road. 3 datasets per record.* R*, the number assigned to this road, of type* I^+*.* i*, the number of the junction at one end of this road, of type* I^+*.* j*, the number of the junction at the other end of this road, of type* I^+*.*

Data structure 3. *Cluster.* C*, the number assigned to this cluster, of type* I^+*;* n*, the amount of roads belonging to the cluster, of type* I^+*;* f*, the fractal dimension of the cluster,* R^+*;* M*, set of member roads of this cluster,* $M = \{m_i : m_i \in (\text{cluster } C), m_i \in R, i = 1, \cdots, n\}$*.*

For illustrative purpose a network of an imaginary city called City X is considered in Figure 2 and Figure 3. Figure 3 is that of the networks in Figure 2 after the construction of a ring road. The percolation probability of the networks shown in Figure 2 was found to be approximately 0.75, averaged over 20 simulations.

The percolation probability of the networks shown in Figure 3 was found to be approximately 0.72, averaged over 20 simulations. That is the addition of the ring road reduced the p_c.

The results shows that the ring road added resulted in the reduction of the p_c. And since the p_c of this network is > 0.5, this also resulted in the reduction of the probability that this network will be in a congested situation.

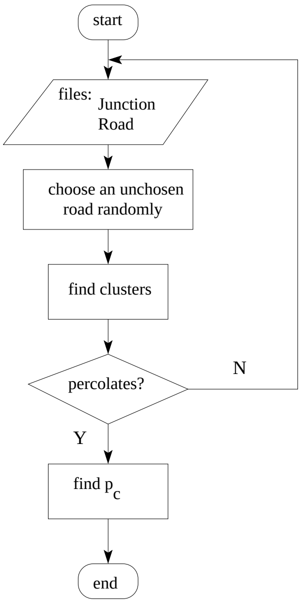

Figure 1.. *Procedure for findig the critical parameter of a traffic networks.*

Traffic control

Let S be the situation where the percolation of roads where traffics are flowing occurs. S is namely *space percolation.* And let H be the situation where the percolation of congested roads occurs. H is namely *car percolation.* The decisional cases in Figure 4 are cases where $(S \wedge \neg H)$, $(S \wedge H)$, $(\neg S \wedge \neg H)$ and $(\neg S \wedge \wedge H)$ for cases number 1, 2 and 3 respectively. The parameter t_k in Figure 4 is the sampling time of the data. t_k could be approximately 5 minutes.

Conclusion

I hope that this work has shown the possibility of applying the percolation theory to modelling and control problems of traffic networks. More work is still needed to be done in order to deliberate upon this idea. It is important to note also that this approach, if turned out to be feasible, could be similarly applied to fields where other kinds of traffic are concerned too, to mention but one of such fields is the traffic inside the Internet.

[Figure 2 used to be here.]

Figure 2. *Diagram representing a traffical networks of* City X.

[Figure 3 used to be here.]

Figure 3 *Diagram representing a traffical networks of* City X *after the addition of a ringroad.*

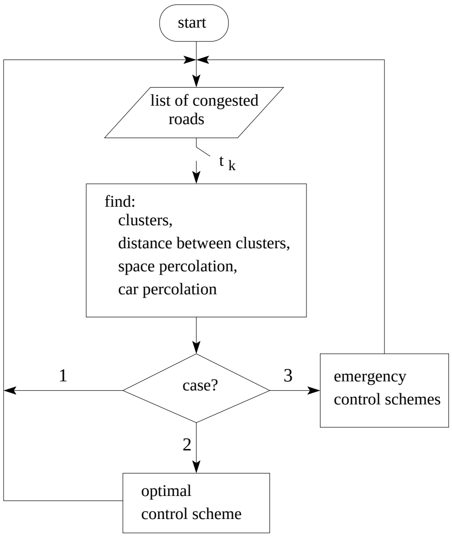

Figure 4.. *Traffic control scheme for* City X.

§ E.22 Variable structure control and singular perturbation

Variable structure control for a singularly perturbed system‡

Kittisak Tiyapan†

Abstract

A case study of a singularly perturbed system has been studied. A variable structure control scheme was applied to this system. Simulation has been done and the result obtained discussed.

1. Objective

The objectives of this paper are

1 . to study some characteristics of a sign function input,
2 . to study the effect of ε in a singularly perturbed system, and
3 . to study the effect of ε in a singularly perturbed system having a VS [Variable Structure] feedback.

2. Introduction

This work is based on the system studied and described in my previous work (Tiyapan, 1998). It is the extension to the work and will be a part of the result in the 2^{nd} edition of the technical report.

3. The system

Consider the following system

$$\dot{x} = E^{-1}Ax + E^{-1}Bu \tag{1}$$
$$y = Cx + Du. \tag{2}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -8 & -5 & 1 & 1.25 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & -11 & -2.5 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 2.82843 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} -2.82843 & 1.59099 & 0.70711 & 0.88388 \end{bmatrix} \quad \text{and} \quad D = 0.$$

4. Simulation on the system

The following figures are the plots of the system without a control input.

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[Figure 1 used to be here.]

Figure 1 *Root locus when (a)* $\varepsilon = 1$, *(b)* $\varepsilon = 0.5$, *(c)* $\varepsilon = 0.1$, *and (d)* $\varepsilon = 0.001$

[Figure 2 used to be here.]

Figure 2 *State diagrams of a system with* $\varepsilon = 1$ *(a)* $x_1 - x_2$, *(b)* $x_1 - x_3$, *(c)* $x_1 - x_4$, *(d)* $x_2 - x_3$, *(e)* $x_2 - x_4$, *and (f)* $x_3 - x_4$.

Figure 2 is the state plane for $\varepsilon = 1$. The initial conditions were taken as Initials 1.

Initials 1 is the set of points

```
for $i = -5, 5$
    for $j = -5, -3, -1, 1, 3, 5$
        for $(p,q) = (1,2), (1,3), (1,4), (2,3), (2,4), (3,4)$
            $x_p(0) = i$
            $x_q(0) = j$
            $x_k(0) = 0$, for all $k$ which are not equal to $p$ or $q$
        endfor
    endfor
endfor
```

□

Figure 3 shows the state planes obtained when $\varepsilon = 0.01$. At higher values of ε the oscillation becomes more prominent (due to limitted space examples such as state planes when $\varepsilon = 10^{-4}$ will be included in the 2^{nd} edition of Technical Report #1 (Tiyapan, 1998)).

[Figure 3 used to be here.]

Figure 3 *State diagrams of a system with* $\varepsilon = 0.01$ *(a)* $x_1 - x_2$, *(b)* $x_1 - x_3$, *(c)* $x_1 - x_4$, *(d)* $x_2 - x_3$, *(e)* $x_2 - x_4$, *and (f)* $x_3 - x_4$.

[Figure 4 used to be here.]

Figure 4 *State diagrams of a system with* $\varepsilon = 1$ *(a)* $x_1 - x_2$, *(b)* $x_1 - x_3$, *(c)* $x_1 - x_4$, *(d)* $x_2 - x_3$, *(e)* $x_2 - x_4$, *and (f)* $x_3 - x_4$.

[Figure 5 used to be here.]

Figure 5 *State diagrams of a system with* $\varepsilon = 0.01$ *(a)* $x_1 - x_2$, *(b)* $x_1 - x_3$, *(c)* $x_1 - x_4$, *(d)* $x_2 - x_3$, *(e)* $x_2 - x_4$, *and (f)* $x_3 - x_4$.

The root locus plots of this function are shown in Figure 1 (a), (b), (c), and (d) for $\varepsilon = 1, 0.5, 0.1, 0.001$ respectively. The transfer functions

$$T(s) = \frac{2(s+5)(s^2+4s+29)}{(s+10)(s+1)(s^2+8s+20)}, \tag{3}$$

$$T(s) = \frac{2(s+10)(s^2+4s+29)}{(s^2+8s+20)(s+2.2984)(s+8.7015)}, \tag{4}$$

$$T(s) = \frac{2s^3+108s^2+458s+2900}{(s^2+8s+20)(s^2+11s+100)}, \tag{5}$$

$$T(s) = \frac{2s^3+10008s^2+40058s+(2.9\times10^5)}{s^4+19s^3+10108s^2+80220s+(2\times10^5)} \tag{6}$$

are for the case of Figure 1 (a), (b), (c), and (d) respectively.

5. How ε affects the system characteristics

Figures 6, 7 and 8 show the Bode-, Nyquist- and Nichols plots of the system respectively, with various values of ε.

[Figure 5 used to be here]

Figure 5 *State diagram of a feedback system with* $\varepsilon = 0.01$ (a) x_1-x_2,(b) x_1-x_3,(c) x_1-x_4,(d) x_2-x_3,(e) x_2-x_4,(f) x_3-x_4.

[Figure 6 used to be here.]

Figure 6 *Bode plot (*$\varepsilon = 1$ *as a solid line,* $\varepsilon = 0.001$ *a dashed line,* $\varepsilon = 10^{-5}$ *a dashdot line, both* $\varepsilon = 10^{-7}$ *and* $\varepsilon = 10^{-9}$ *as dotted lines)*

[Figure 7 used to be here.]

Figure 7 *Nuquis plot (*$\varepsilon = 1$ *as a solid line,* $\varepsilon = 0.5$ *a dashed line,* $\varepsilon = 0.1$ *a dashdot line, both* $\varepsilon = 0.01$ *and* $\varepsilon = 0.001$ *as dotted lines)*

[Figure 8 used to be here.]

Figure 8 *Nichols plot ($\varepsilon = 1$ as a solid line, $\varepsilon = 0.001$ a dashed line, $\varepsilon = 10^{-5}$ a dashdot line, $\varepsilon = 10^{-7}$ as circle- and $\varepsilon = 10^{-9}$ x-marks)*

6. The control

Use the feedback scheme

$$u = Cx + \operatorname{sgn}(CE^{-1}Ax). \tag{7}$$

where

$$\operatorname{sgn}(f) = \begin{cases} 1 & ,f > 0 \\ 0 & ,f = 0 \\ -1 & ,f < 0 \end{cases}$$

7. Simulation on the controlled system

When $\varepsilon = 1$ the result can be shown as in Figure 4. Figure 4 is quite similar to Figure 2 where the system was without any control and with the same $\varepsilon = 1$. Here the initial conditions used were the same as those used for Figure 2.

The next picture, Figure 5, shows the result of a simulation when $\varepsilon = 0.01$. Figure 5 should be compared with Figure 3 which is the case with the same value of ε ($\varepsilon = 0.01$), but which has no control input. Both Figure 5 and Figure 3 has the same set of initial conditions.

8. Discussion

The only equilibrium point is (by setting $Ax = 0$) at the origin regardless of the values of ε. Two of the eigenvalues are affected by ε while the other two are not. The four eigenvalues are $-4 \pm \jmath 2$, and $-11/2 \pm (121 - 40\varepsilon)^{1/2}/2$. The two poles which are not affected by the value of ε (the ones located at $-4 \pm 2\imath$) have the natural frequency $\omega_n = 4.4721$ and the damping factor 0.8944.

$\omega_n \to \infty$ as $\varepsilon \to 0$ while $\zeta \to 0$ as $\varepsilon \to 0$ (Tiyapan, 1998). Figure 6 shows that ω_n increases as ε decreases.

The Nyquist plot in Figure 7 shows that the system is always stable since (see for example Ogata, 1970) $Z = N = P = 0$ satisfies $Z = N + P$, where Z is the number of zeros of $1 + G(s)H(s)$ in the right-half s plane, N the number of clockwise encirclements of the point $-1 + \jmath 0$ and P the number of poles of $G(s)H(s)$ in the right-half s plane.

The root locus plots in Figure 1 (a) to (d) show that the system is minimum phase for all $\varepsilon > 0$. Figure 6 also shows that this is true because the phase (see for example Ogata, 1970) $\phi \to \deg -90(q - p) = \deg -90(4 - 3) = \deg -90$ as $\omega \to \infty$, and the log-magnitude curve is $-20(q - p) = -20(4 - 3) = -20$ db/decade, as $\omega \to \infty$. Here p and q are the degrees of the numerator and denominator polynomials of the transfer function respectively.

Together with Equation 3 to 6, Figure 1 shows that two complex zeros and two complex poles remain unmoved when ε changes. As ε decreases the zero on the real axis moves towards $-\infty$. When it passes the pole on its left hand side the shape of the root locus changes. The two poles on the real axis moves toward each other. They meet at -5.5 and then become complex conjugate poles. This also changes the shape of the root locus.

The Nichols plot (Figure 8) show that at a given value of gain K the peak of resonance value of M, $M_r = |G/(1 + G)|$ increases as ε decreases.

Figure 2 to 5 shows how the states oscillate as ε decreases (the frequency of oscillation increases as ε decreases). The state variable x_4 which was affected directly by ε was the one which fluctuates the most (the magnitude of fluctuation increases as ε dereases). Rougly speaking, the trajectory changes shape and direction (perhaps due to the change in the orientation of the hyperplane). The trajectory of the system with VS feedback oscillates around the equilibrium point.

9. Miscellaneous

Simulation in this work was cone on Simulink Version 2.1 in Matlab Version 5.1.0.421 on Sol2. The rest of the result of this study, as well as related studies, will be included in the 2^{nd} version of Technical Report # 1. I thank Professor K. Furuta for suggesting this interesting topic.

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§ E.23 Projects on translation

Expression, Transliteration and Word-creation

a Ph.D. Research Project

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Introduction

This report is divided into three main sections corresponding to three problems it concentrates upon, namely expression, transliteration and word creation. *Expression* represents problems in translation, *transliteration* represents attempts to find one-to-one correspondences in translation and *word creation* represents creative processes found in translation. Each of these three sections has a review section of its own literature review which is only given briefly here. There is also Section 1 which gives an overall review of all the things all the three sections are said to represent. This is to give some direction for future researchers who would like to do a research in related area.

As one can never avoid computers now in the 21^{st} Century I have also dedicated Section 2.2 for whatever contribution I may manage to make electronically in the future, for the benefit of future researchers.

1. Review of literature

The information in this section is mutually exclusive to those in Section 1, Section 4.1 and Section 5.1 to follows and must be put together to make a complete picture of my researching related literatures. Section 2.1 gives another review for any reader with an engineering mind.

Problems in translation come in many disguises. One always find classic examples browsing through literature on translation. For example, Michael Schreiber gives one example of a poem where rhyming is essentially transliterated from German into English (Schreiber, 1993, page 146.)

2. Methodology and tools

2.1. Review of literature: Tools

There have been various attempts to model the translation process (for example Bell, 1991, page 59.) This type of modelling is in line with Linguistics which is the attempt to describe grammars using Mathematics. There are books written mathematically about languages (Wood, 1993; Arntz, 1982—this one not too mathematical but uses tools in Mathematics nicely to organize ideas) but they tend not to address all important issues like metaphors and expressions. However, with Mathematics going more and more towards nonlinearity, seeking order in what is seemingly disorder as in the theory of Chaos, merging itself with nature artistically as in the theory of Fractals, or philosophically as in the theory of Percolation, one can now hope one day to understand, for example, the working of a translated metaphor. Having said that, one has to admit that one does find impossibilities even in Mathematics itself, for example division by zero or the crossing of the Königsberg's bridge. So one could safely say that one can rest assured *untranslatability* (Bassnett-McGuire, 1980, page 32) will always stay in sight.

2.2. Computer-related things

LaTeX will be used for writing reports. Metafonts which comes with TEXwill be used for creating fonts for those languages which still has no electronic standard for processing and archiving. Creating a new standard is done among Internet community using the Request For Comments protocol or RFC. For the purpose of report writing, the necessary standards would be made. This would become an additional resource which benefits Internet users as the whole.

2.3. Languages

Languages used in examples and case studies are those which I have an interest in; at this stage I set no limit to their number. There will be some unavoidable problem when one studies languages. A person normally has only one mother tongue; and mine is actually not even Thai but Lanna. The problem lies in the fact that one use a language in studying languages. So the study could not become totally free from the influence of that language. I will try to minimize, and if possible find a good way to minimize, this problem. But, as Prasad has said that *an Indian writer should not write like a British writer* (Prasad, 1999), this may not necessarily be a disadvantage.

3. Expression

Translation of expressions has got everything to do with the question of *translatability*, namely what is translatable and what is untranslatable. Expressions often are metaphoric and cutural in nature. One could argue with reason that expressions could be best translated with a mentality of a person who is very much at home with both the source and the target languages.

3.1. Review of literature: Expression

Translation of *expressions* depends directly upon one's idea about what things are equivalent. Mona Baker deals with *equivalence* in various forms, as well as *faux amis* Baker, 1995.

Unless one stops reading information and ideas never stop coming in. Not only are there good translated classics to refer to, but also there are examples written by educator-translators (for example Snell, 1971).

3.2. My contribution

It seems unlikely that an equivalent expression could be created by rendering word for word one language into another. Among other things, I would like to look at criteria for good equivalence in the translation of expressions.

The Lanna language still exists as a dialect of Thai though it used to be an independent language of South East Asia in the 14^{th} Century. Due to a political reason its scripts had been put out of use for nearly a century, but is now reviving. The language is no longer written, or at best is transliterated poorly using Thai alphabets. But the language is very rich in expressions. I would like to draw some example for this section from this language which is my mother tongue. This is one of the reasons of my mentioning the RFC standards [in Section 2.2].

4. Transliteration

Transliteration of words across languages is sometimes difficult because sounds in one language seldom correspond to those in another in one-to-one manner. This problem is confounded further as the number of alphabets and vowels usually differs, and there are cases where there are more than one alphabets having the same sound. One can not ignore transliterations, neither can one take them for granted. They are not only useful for turning proper names from one language to another; they are also useful when one want to adopt words from foreign languages. Any convention in transliteration is for the two languages concerned only. It would be nice to have a *universal* convention which can be used among variety of languages, but I doubt the usefulness and user-friendliness of such a convention if it exists.

4.1. Review of literature: Transliteration

Literature to be researched covers sources which ranges from organizational codes of practice in transliteration to transliterated material within the body of published works. There are some standards which are already very good, for example that for translating Mandarin into English called pīn yīn. But unfortunately, or fortunately for that matter, most existing standards still leave much room to hope for.

4.2. My contribution

Transliteration of scripts between language pairs will be studied. Guidelines, procedures, or if possible standards are the aim.

5. Word creation

There are many instances where one needs to create a new word. When anything new is invented in one country, taking them abroad means finding a name that it could be called there. One may choose either to adopt the new word from foreign origin, or to make up a new word in one's own tongue for the new thing. The first method is called transliteration and has already been mentioned in Section 4. The second one normally has to resort back to the infrastructure of each language before the new word can be created. This infrastructure is the underlying structure or the building blocks of the language. Ancient languages where root of the words of those languages belong are examples of such structure. For most of the languages in Europe these languages are Greek and Latin. For many languages of South Asia they are Pali and Sanskrit. Again, this structure could be symbolic, for example the pictorial nature of the Chinese characters which influences those other languages that use them, namely Japanese and Korean.

5.1. Review of literature: Word creation

To be done. Some of the material here might come from what I have translated myself as a translator, most of which have not been published.

My contribution

I would try to find a procedure for translating scientific terms. And then I would translate those terms which has never been translated before. These terms would cover other fields as well as that of Control Systems where I have some.

Other than scientific terms there would be jargons from a variety of disciplines to consider, ranging from the idioms used in ancient Weaponries to terms used in Music. In whichever field considered, I will try to cover as broad a spectrum of cultures as possible; weaponry jargons of Polovtsy will be studied as well as those of Ayudhya.

Usefulness of this project

Some of the merits of this project would be,

- Examples of problems and their solutions given would become a resource for translator as well as collection items for lovers of both the source and the target language.
- Standards created for archiving and writing scripts in the reports for this project as mentioned in Section 2.2 would become a resource on the Internet for anyone to use.
- Scientific terms and the procedure for producing them could become useful.

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□

The use of transliteration, equivalence and coinage in translating the metaphor of literature and scientific technology and jargon in the languages of East Asia

a Ph.D. project proposal to UMIST

Kittisak Nui Tiyapan

Introduction

Translation between the languages of Europe and those of East Asia is interesting as well as poses many difficulties. One of the reasons for those difficulties is that these languages of the two continents are from different family. For a similar reason the translation of the rest of European languages into or from Hungarian for example, which belongs to a different language family‡, is equally interesting. So one could perhaps say that the difference has more to do with the difference in the historical origins of the languages than it has to do with differences in the geographical locations or the ethnic difference.

Objective

These are the main objectives

- To understand how one could coin up words from science of a foreign origin.
- To be able to coin words which best represent these new terms.

Methodology

The methods used for this study will start from the *analysis* of standards and conventions, both existing ones and those in the past. I will attempt to *synthesize* systematically in order to gain the whole picture. *Refutation* of some of the existing practices may become necessary at this stage. In other words, everything will be put together and slack practices commented, improved upon or changed.

Examples

The following are some examples of technical words in different languages [2].

signum. which means a function that has the value of +1 for all independant variables greater than zero, 0 when independent variable is zero, and -1 for those less than zero. In Italian it is called *funzione segno*, in French *fonction signe.* The Spanish word for this term is the same as the English one, and so are the German and Dutch terms for it.

Resonance. is called *resonancia* in Spanish, *risonanza* in Italian, *résonance* in French, *rezonans* in Polish, *résonans* in Swedish and *resonantie* in Dutch. In Thai, however, instead of being transliterated it is called *Kan Kamton*, a term which has been successful in the sense that it has gained acceptance in learned societies and has obtained the same *mental image* of *resonance.*

parameter. in the same in German, Spanish and Dutch. In Thai it is used in a transliterated form, and so is in Japanese.

[1] Hilary Putnam. Mind, language and reality. *Philosophical Papers.* **2**. Cambridge University Press. 1975.
[2] *Electricity, Electronics and Telecommunications. Multilingual Dictionary.* International Electrotechnical Commission, Elsevier. 1992.
[3] `http://www.britanica.com`

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‡ Hungarian belongs to the Ugric branch of the Finno-Ugric language family. Other members belonging to the same branch are Mansi(Volgul) and Khanty (Ostyak) which are both spoken in Western Siberia [3]

§ E.24 Abstracts of books I wrote

Interesting English

This book is written in Thai with the title *Bhạṣạ angkṛiṣ an nȧṣoncaị* during the time of economic gloom for both myself and my country Thailand (*cf* 10.24). It is built around three major experiences which I consider to be the turning points of my life, namely my studying the Sixth Form English in New Zealand (1983), my acquaintance with a british scholar Amnad Khitapanna since 1985, and my experience in Budapest and Europe (1991). In 104 pages I try to pass on as much as many things as possible those things which I consider valuable from all the three sources mentioned. With my teaching a class in English Literature at the Pradíṣṫḥ Center in Bangkok as a catalyst the book was published at the Chulalongkorn University Press (2000) under the trademark *Kittix* which has now become my own. From my study of the Sixth Form Engish comes the sections on false friends, alliteration and assonance, onomatopoeia, jargons, Macbeth, several poetic quotations, and the section on books and movies, from discussions with Amnad the seemingly prefixed words for example *dishevelled* and *unkempt*, and songs by Andrew Lloyd Webber, and from my experience of working in Eastern Europe during 1991 the game *twenty questions*. The rest of the materials are mainly what I used in teaching the English class mentioned. They are tragedies *vs* comedies, Titus Andronicus, Julius Caesar, Romeo and Juliet, poems by Don Marquis, and homophones. Apart from that this book also mentions some word games the examples of which are riddles, aeiou-ordered words (for example, *abstemious*), spoonerisms, tongue-twisters, anagrams, scrabbles, crosswords, Targets, doublets, syzygies, word squares, and acrostics. There is a figure which shows the relative degrees of adjectives on a straight line, and also a Venn diagram showing the comparative domains of the English words *wok* and *pan*, and the Thai word *kada*. I had no laser printers and camera-ready copy of the book was printed from a laser printer at the Control System Engineering group at the Electrical Engineering Department, Chulalongkorn University with the courtesy of my former supervisor Assistant Professor Watharapong Khovidhungij [Wacȧrȧbongṣ́ Ḳhọviḍụrkic] whom I deeply thank. But the quality of the printing is not very good, which results in some parts of the book being difficult to read. To answer the question I have been asked, the picture in the biography section was from the plastic badge which I used to wear when working at Jasmine in Bangkok.

Free translation of English

The title of the book written above is the English translation of the actual title *Plạe kled́ Angkṛiṣ*, the word *plạe kled́* being an adaptation from the jargon of the old thai sword fighting *f̣an kled́* which means *an improvising practice*, or to do this when the word is a verb. Again the camera-ready copy being printed at the Control System Engineering Department except this time with the courtesy of Dr. Manop Wongsaisuwan who is my senior both at the Electrical Engineering Department, Chulalongkorn Unversity, as well as at the Furuta Laboratory, Tokyo Institute of Technology, whom I consider a wizard in the computational matters, and to whom I thank. This is a book about translation written from my experience in translating and inspired by my teaching in a translating class at the same Pradíṣḋḥ Center already mentioned. It is a book of translation both from Thai into English as well as from English into Thai. As such, none of the various examples given is easy and straight forward, and all of them have baffled some translators in the past. It was written at the height of my artistic and literarical ability and at the time when I also enrolled in and attended classes in European (English included) literatures at the Ramkamhaeng University in Bangkok. The book starts off with the phrase *kạrplạe gụe kạrplạe gụe kạrplạe*, literally *translation is translation is translation* but which I deliberately translated as *to write is to write is to write* all languages and writings being but translations of the thought. Historically the romans seldom translated but used Greek in their writings instead, and the greeks never did for theirs were the originals. The book talks about creativity in translation and plagiarism. In general rules may be broken at the right time. Styles of writing are mentioned again without repetition of my previous book. Namely these are the registers and modes of use, alliterations, assonances, onomatopoeias, metaphor, simile, repetition, coupling, rhetoric, allusions, direct and indirect speeches, mythological materials, burlesques, comparative usages of punctuation marks of both languages. The books says that simplicity is the mother of invention. Archaism and parallellism are mentioned, and so are jargons from various disciplines. Examples from literature includes those by Shakespeare, Lewis Caroll, Yeats, and Sundaurbhụ̀. In particular the inventive usages of Cockney English is problematic in translation, no negative pronouns exists in Thai so one needs to try hard to find their equivalences. Some of the jargons I mentioned are those of thai arts, fruits, spices, illnesses, anatomy, fauna, and flora. Some of the humourous clippings from the internet obtained from my friend and computer professional Ken Labinjok in Bristol are also shared with the readers in Thai, some of which needed some explanation to become funny. As for the rest, most of the things I wish to say the book has hardly left out; I would never have thought that I had put so many of them in these 106 pages had I not looked at it again just now. By the way, does anyone happen to know that *novemdecillion* in American is 10^{60} but in English is 10^{114} I wonder. Both my first book and my second one have one problem alike, that is that the computer that I use is of japanese make and can fluently write Japanese but not Thai. One problem which persisted and baffled me throughout is the mysterious disappearance of all the *ỵ*'s. Ironically enough the alphabet *ỵ* is in Thai called *yạu phụ̀ỵing*, literally *w for women*! The page on biography is a little more comprehensive than that of the first book. Having mentioned a picture in the previous one, the picture in this book was taken in front of my house in Bangkok.

Voronoi Translated‡

That succinct title was followed by a more descriptive one, *Introduction to Voronoi tessellation and essays by G. L. Dirichlet and G. F. Voronoi*, which summarises this book. This is the first book, and hopefully the last one, I have written which has a grey cover, *grey* being the symbol of *Gandalf the Great* which in turn rhymes with *Graham Davies* the name of my supervisor and sponsorfrom end 2000. It is also the first book which I typeset with TeX which I consider complicated but works, while both of the previous two books were typeset using LaTeX which I consider simple and works, the reason for the change being the upgrade of the latter to LaTeX2ε which I consider complicated and (presumeably) works together with the fact that it has TeX working in the background. The switching over thus helps reduce one shell around the equivalences of the only desirable attribute left which is that of *complicated but works.* Here the two venerable alphabets *v* and *n* rules throughout, starting from the title *Voronoi*, the name of a russian mathematician of the nineteenth century, then the picture taken in *Vienna* on the front, and that in *Venice* on the back covers; if you look carefully enough, somewhere in the book you will find another picture of a stone masonry with a caption saying *Verona* which Professor Davies decidedly says is not a Voronoi tessellation. The introduction gives some historical backgrounds and pictures of various orders of covering lattices of the Voronoi tessellation in two dimensions, which can be used to represent for example a realistic picture of the conglomeration of grains within some kind of matrix. The triangular and the hexagonal lattices are examples of dual lattices. The picture of the hexagonal and kagome lattices superimposed on top of each other is not a picture of the Eden Project in Cornwall but an example of covering lattices. This is not a free translation of the seminal works by G. F. Voronoï and G. L. Dirichlet. I tried to retain the original as much as possible, namely where I thought the grammar in the original was incorrect or the structure of the sentence is convoluted I coined up the equivalence, though this is by no mean how I always translate. As a result the translation is necessarily more difficult to read than it would have been had it been freely summarised. My reason for doing it in this style is firstly in order to transfer as much nuances across as possible, and secondly simply because I am not a mathematician and therefore in no position to summarise the work of one. Last but not least must be my own shortcoming and inexperience in translating from French and German. The preface of the book gives a brief history of the *Journal für die reine und angewandte Mathematik* while the introduction that of the Voronoi tessellation. Also one knows from the latter that the name *kagome* is a Japanese word which means *basket interstice or pattern*, the fact that I realised while reading the Japanese language in Tokyo. *Me* is the word for *eye* as well as *pattern*, and in Japnanese *kago* means *an intersticed basket* as much as *taklạ̀* does in Thai. There is hardly discipline in which Voronoï tessellation has not found applications and the list of relevant areas is endless, ranging from management to computation to physics. This book has 282 pages of contents.

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‡ ISBN 974-13-1503-1

F

§ F. Translation

§ F.1 G. L. Dirichlet, 1848

On the reduction of positive quadratic form with three indeterminate integers.

([Lecture in physical- mathematical class meeting of the academy, on 31^{st} *July*, 1848 †]), [by G. L. Dirichlet]
[translated by K N Tiyapan]

It is well known that Lagrange had pointed out for the first time that every binary quadratic form reduces, ie. can transform into another equivalent one the coefficients of which satisfy certain inequality conditions, and at the same time had proven that in every class of positive forms there always exists only one such form, so that in this case the various values of a given determinant corresponding to reduced forms can serve as the representatives of the different classes. Later on after in the "Disquisitiones arithmaticae" the ternary form were looked at from a general point of view did it become necessary for the further development of this theory to extend the study for the positive binary forms by Lagrange to the ternary ones, ie. to find out such inequality conditions between the coefficients that would satisfy one and only form in all classes.

This expansion linked with great difficulties is achieved by *Seeber* in a work specifically devoted to the positive ternary forms, the principal contents of which settles it and which *Gauss* characterises in a most interesting announcement † as follows:

> *We must do full justice to the spirit of the thoroughness by which these facing ‡ us have gone through, and when we for all that have to feel sorry that a great and perhaps much discouragingly complicated nature is attached to it, here the solution of the problem takes 41 pages and the proof 91 pages, thus we will see this by no means as a respected criticism. If a difficult problem or theorem to solve or to prove exists, then the first seeming idea is always to be recognised as a step that a solution or a proof has been found after all, and the question whether this were not of an easier and simpler way would be possible as long as in so doing such a futile question is not considered as of practicability. Therefore we look upon it as untimely to dwell on this question.*

The great complication of Seeberian method has for a longtime stimulated me to set up the theory of reduced ternary forms by a simpler method. As I now allow myself to communicate to the class the result of my effort directed towards this, I think in the interest of briefness and, if I could say so, of the lucidity of the presentation, to have to abide by the geometrical form, in which I have conducted the investigation to which I have laid down as basis the noteworthy relations which occur among the quadratics with two or three elements and with known spatial forms. I begin with the explanation of the outline already given by Gauss in the mentioned announcement on these relations.

§1

The ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy = \varphi, \tag{1}$$

in which we regard x, y, z as first, second, third element, called positive when φ does not become negative for real values of these elements; in one such form the coefficients:

$$a, b, c$$

are always positive, while the coefficient combinations:

$$a'^2 - bc, b'^2 - ac, c'^2 - ab, aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c' = -D, \tag{2}$$

the last $-D$ of which is called the determinant of the form, are negative. † Owing to these conditions there are always three through the equations:

$$\cos\lambda = \frac{a'}{\sqrt{bc}}, \cos\mu = \frac{b'}{\sqrt{ac}}, \cos\nu = \frac{c'}{\sqrt{ab}}$$

fully determined acute or obtuse angles λ, μ, ν, from which a three-edged corner can be built, here the condition necessary to this:

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu - 2\cos\lambda\cos\mu\cos\nu < 1$$

with $D > 0$ coincided. Here nevertheless with the same angles λ, μ, ν two corners symmetrical to each other could be built, therefore we will agree to always choose the corner from these two, with which the edges, as they lie opposite to these angles in sequence, follow one another from left to right with regard to a straight line directed from the vertex 0 to the inside of the corner which can be thought of as going upward. If we now look at the three edges as the positive axes of a coordinate system we could connect the entire infinite space with our form in which we view the product $x\sqrt{a}, y\sqrt{b}, z\sqrt{c}$ as the coordinates of an arbitrary point of the space, and then φ expresses the square of the distance of this point from the vertex, or more general still the square of the distance of two points, the correspondent coordinates of which have those products to differences.

If one establishes now with three new indeterminate elements x', y', z' the linear expressions:

$$x = \alpha x' + \beta y' + \gamma z', y = \alpha' x' + \beta' y' + \gamma' z', z = \alpha'' x' + \beta'' y' + \gamma'' z', \tag{3}$$

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† Crelle's Journal, V. 20, p. 312

† Disquisitiones arithmeticae, art. 271

of which only one restriction shall take place, that the determinant set up from the 9 coefficients α, β, γ, α', β', γ', α'', β'', γ'':

$$\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma'' = E \tag{4}$$

is not zero, then φ changes into a new form φ', with regard to which all correspondings shall be indicated with the accented alphabets. If one lets the new form again correspond to an infinite space, then through it two infinite spaces connect point for point with each other, while every two points correspond to each other when in the expressions of their coordinates:

$$x\sqrt{a}, y\sqrt{b}, z\sqrt{c};\quad x'\sqrt{a'}, y'\sqrt{b'}, z'\sqrt{c'}$$

the elements x, y, z and x', y', z' are linked with one another through the equation (3). If the expressions just written are the coordinate differences for two pairs of corresponding points, then apparently the same relation among $x, y, \ldots$ still holds, out of which from the above and as a result of $\varphi = \varphi'$ it follows immediately that the distance of every two points of a space is equal to the distance of corresponding ones of another. The two spaces connected with each other are therefore either congruent or symmetrical, ie. they can, while the beginning points 0 and 0′ are laid on each other, come to such a position that either every point falls on its correspooonding one or on the opposite point of the latter, when we call for short opposite points two points of the same space which lie from the beginning point at the same distance and in the opposite direction. In order to decide which of these two cases takes place, one has lines to draw in the one space from the vertex to three arbitrary points, and then to investigate whether the straight lines drawn in the other from its vertex to the corresponding points present a corresponding series or the opposite one. If one takes for example in the second space the lines from the points with the coordinates:

$$\sqrt{a'}, 0, 0;\ 0, \sqrt{b'}, 0;\ 0, 0, \sqrt{c'}$$

drawn, lines falling on the positive axes of the second space, then these follow one another from the agreement dealt with above from right to left. For the corresponding points in the first space one has the coordinates:

$$\alpha\sqrt{a'}, \alpha'\sqrt{a'}, \alpha''\sqrt{a'};\ \beta\sqrt{b'}, \beta'\sqrt{b'}, \beta''\sqrt{b'};\ \gamma\sqrt{c'}, \gamma'\sqrt{c'}, \gamma''\sqrt{c'}.$$

In order to determine whether the lines directed to these points follow one another from left to right, ie. as the axes of the first space, or follow in the reverse order, one can make use of the theorem which is known or easily derivable from known properties ‡, from which the straight lines drawn to the three points $(\xi, \eta, \zeta), (\xi', \eta', \zeta'), (\xi'', \eta'', \zeta'')$ present the same series as the axes of ξ, η, ζ or the opposite one, according to the determinant built from the 9 coordinates, when one gives the term $\xi\eta'\zeta''$ in it the positive sign, is positive or negative. For our case this determinant becomes $E\sqrt{a'b'c'}$; therefore congruence symmetry holds according as E is positive or negative.

Til now the elements x, y, z had arbitrary values. If we let them now only further mean integers, then instead of the integral space we have an infinite system of points parallelly arranged, ie. a point system of which through the intersections of three lines parallel equidistant planes would be created. If we assume now further that the substitution coefficients α, β, γ are also integers and E has the values ± 1, then every integral combination x, y, z would represent an integral combination x', x', z' and vice versa. The parallelepipedal systems thus connected with one another would as a result coincides with the other or with the opposite points of the latter. Yet, here the opposite points of points of one such system again make the same system, the two cases are not different from each other, and this becomes evident also from the circumstance that φ' remains unchanged when one takes α, β, γ with opposite signs through which E changes into $-E$. The two systems are therefore always congruent, and one sees that systems which correspond to two equivalent ternary forms φ and φ' are the same spatial structure in two different patterns. Conversely equivalent forms represent any two different parallelepipedal patterns of the same system. If one takes namely any one point of the system as the common starting point, then one has between the coordinates relation to the two axis systems and therefore also between the elements x, y, z, x', y', z' proportional to them linear equations without constant term, ie. equations of the form (3), and here from our supposition, when x, y, z are integers, x', y', z' must also have the same characteristic and vice versa, therefore it follows that $\alpha, \beta, \gamma, \ldots$ are likewise integers and that $E = \pm 1$. On the other hand one has for the homogeneous entire values of the elements the equation $\varphi = \varphi'$, which accordingly also identically takes place, q.e.d.

Similar interrelations occur between a positive binary form:

$$lx^2 + 2mxy + ny^2$$

and a system of points parallelogrammatically arranged. One takes here two axes leant against each other under the angle θ determined through the equation $m = \sqrt{ln}\cos\theta$, while one always invariably proceed with the discrimination of these axes and for example chooses the second on the left-hand side of the first one, after a fixed side of the plane is denoted as the higher one and $x\sqrt{l}, y\sqrt{l}$ viewed as coordinates, one would obtain a system of points completely determined through the quadratic form, which could be considered as the intersection of two series of equidistant parallel lines. If then between two forms the so-called proper equivalence takes place, so that $\alpha\delta - \beta\gamma$ in the substitution equations $x = \alpha x' + \beta y', y = \gamma x' + \delta y'$ is equal to the positive unity, then the corresponding systems can be brought to the coincidence through movement in the plane, while in the other case where $\alpha\delta - \beta\gamma = -1$, to say in general, one of the systems must be shifted for this purpose.

§2

After we have established in the foregoing the connection between the quadratic forms and certain geometrical patterns, there are a few further properties of these patterns to develop, whereby we would for short call a system of points arranged parallelogrammatically or parallelepipedally a *system of second* or *third order*, and infinite series of equidistant points in straight line a *system of first order*.

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‡ Disquisitiones generales circa superficies curvas auctore, C. F. Gauss §2. VII

It seems that the common character of all three types of the system consists in that when such a system is brought into another position through a movement without rotation, which we wish to know a displacement of, that a point of it changes into the position occupied by another in the beginning, the same happens for all points, therefore that the system in its new position fully coincides with the system in the original one. It can be easily proved that the movability just discussed completely characterises all three types of the systems, and that a system endowed with this characteristic, when it lies in a plane and contains three points not lying in a straight line such that finally a system contains points at least four of which are not found in a plane, will be respectively a system of first, second or third order.

If one has for example a system of points which lie all together in the same straight line, and a and a' are two adjacent points of it, then through a displacement through which a gets to a', a' would get to a'' which is as far from a' as a' is far apart from a; the point a'' therefore also belongs to the system, and the system has no point between a' and a'', here one such point would be known before the movement between a and a'. Here this inspection could be pursued for both sides in the indeterminate, therefore the assertions is proved.

Now let two adjacent points a and a' be in a planar system with the characteristic feature of movability, so that no point of the system is found in the line aa' between a and a'. Here through the displacement from a to a' the infinite straight line aa' moves along by itself, therefore it follows that the entire points of the system in this straight line makes up a system of first order $\ldots {}''a'aaa'a'' \ldots$. Here then from the assumption the system still has at least one point outside this straight line, therefore let b be one of the points closest to this straight line. If now enters a displacement through which a gets to b, then the system of first order changes into the new position $\ldots {}''b'bbb'b'' \ldots$ and belong in this position to the original system; it is immediately clear that a point of the system can neither be found among the points $\ldots, {}''b, {}'b, b, b', b'', \ldots$ nor among the lines $\ldots {}'bbb' \ldots, \ldots {}'aaa' \ldots$. If one concludes directly, one sees that the entire system can be parallelogrammatically aranged, and that one can choose $aa'b'b$ for a basic parallelogram of it. We further add that through the given construction apparently all parallelogrammatical patterns of which the system is capable could be obtained. It follows from this that the choice of a' up to the obviously necessary restriction that no point lies between a and a' is totally arbitrary, and that b can be taken arbitrarily in the nearest parallel line.

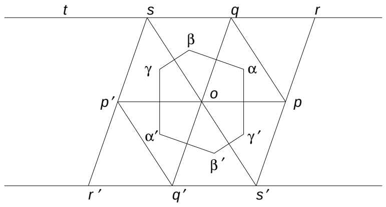
One has finally a system with the characteristic feature of the movability which contains points at least four of which are not lying in the same plane, therefore one lay a plane through any three points of this system not lying in a straight line. Here through any parallel displacement effected with this plane this is moved into itself, consequently points found in this plane build a system of second order from the previous system. As a result one has partitioned this system parallelogrammatically somehow or other, one takes one of the remaining points of the spatial system which lie closest to the plane, and administer a displacement to the system through which an arbitrary point in the plane comes to the point chosen well outside that plane. Through repeated application of this [displacement] and through the movement opposite to it one apparently obtains a parallelepipedal pattern of the given system, and it is immediately clear that the construction specified has the due generality, here the choice of the first plane, the pattern of the system of second order in this plane and finally the choice of points in the neighbouring plane can happen at will.

In the end of this paragraph we will point out further that, as one also partition the same system of second or third order, the parallelogram or parallelepiped lying at the basis of the respective partitioning always retains the same capacity, the geometrical consequence of the sentence is that equivalent forms have the same determinants. If one imagines namely in the plane of a system of second order a line returning to itself, for example a circle line, designates with z the surface area enclosed by it and with s the number of points in the inside of the line, in the course of which it makes no difference whether one wants to include the points on the periphery or not, then obviously the quotient $\frac{z}{s}$ has with growing radius the capacity of a basic parallelogram to the boundary, from which, here s and z are independent of the type of the pattern, the theorem for systems of second order becomes evident. Totally in the same fashion follows the soundness of the assertions for spatial systems.

§3

We will now point out that a system of second order would always admit partitioning by a basic parallelogram, the sides of which are not larger than its diagonals.

I. Let o be an arbitrary point of the system. The remaining points of this system always lie pairwise in the same distance and opposite direction from o. Now let p be one of the points of the pairs, for which the distance from o is smaller than for every other pair. The same smallest distance holds for more than one pair, therefore one would choose p at will in one of these. The given system consists of an infinite quantity of systems of first order congruent among one another and of the same distance, one of which is that for which o and p belong. In one of the two adjacent to this latter one, one takes the point q which is next to o, or, supposing the same shortest distance should occur for two points, arbitrarily takes one of the two. The parallelogram $poqr$ thus obtained has the desired property, here in accordance with the construction $op \leqq oq$, $oq \leqq or$, $oq \leqq os = pq$. A basic parallelogram which satisfies these conditions shall be called a reduced one.



II. We have now the relation between one such parallelogram and the planar system in which it belongs to establish. If $poqr$ is a reduced parallelogram we would be able to, without breaking the generality, assume the angle poq as not obtuse, here in the opposite case the angle at o for the parallelogram adjacent to the same structure is an acute one, and likewise we could assume $op \leqq oq$. Thereupon $or > oq$ is apparent, and we have only the condition $pq \geqq oq$ still to consider. If this is supposed and if we put for the reduction $op = \sqrt{l}$, $oq = \sqrt{n}$, so that consequently $l \leqq n$ the connection of our parallelogram to the entire point system would possibly be described to the effect that the minimum of the distance of any point of the system from

o is equal to $\sqrt{l}$, and that after one has chosen a point at this distance, in all distances still remaining, ie. outside the straight line drawn from o to the former one, the second minimum is equal to $\sqrt{n}$. The precisely stated holds true all in general; what we now add, that namely the first minimum only occurs for the point p (when we always only choose one of two opposite points), the second only for q, holds true with the following exceptions:

1. If $op < oq$, $oq = pq = os$, then the first minimum takes place only for p, the second for q and s.
2. If $op = oq$, $oq < pq = os$, then the minima are equal, and one can exchange p and q with one another.
3. If finally $op = oq = pq = os$, then one can choose one of the points p, q, s as first point and then one of the remainings as second one.

In order to demonstrate the precisely asserted, we have obviously, the opposite points are always equally far from o, only to point out that q lies closer to o, *firstly* than all remaining points in the straight line sqr, with exception of the point s, the distance from o of which according to the assumption is equal to $os = pq \geqq oq$, and *secondly* than all points of the subsequent parallel lines.

Here $pq \geqq op$, $pq \geqq oq$ and the angle poq is not obtuse, therefore the triangle opq and consequently also the oqs congruent with it has no obtuse angle; therefore the perpendicular dropped from o on qs lies between s and q (inclusive), with which
one sets:

$$\cos poq = \frac{m}{\sqrt{ln}},$$

where consequently m is not negative, therefore one has:

$$\overline{pq}^2 = l - 2m + n \geqq \overline{oq}^2 = n,$$

and hence:

$$2m \leqq l, 2m \leqq n, 4m^2 \leqq ln.$$

If one further sets the square of the height of our parallelogram ($op = \sqrt{l}$ regarded as base line) equals k, one obtains for the square Δ of its volume:

$$\Delta = lk = ln - m^2 \geqq \frac{3}{4} ln,$$

and hence:

$$\sqrt{k} \geqq \frac{1}{2}\sqrt{3n}.$$

According to this the second line is at least $\sqrt{3n} = oq\sqrt{3}$ away, and the second point is also established.

III. Here the successive minima $\sqrt{l}, \sqrt{n}$ are decided through the system as such and are independent of any fixed pattern and on the other hand, as we have just seen, correspond in quantity with the sides of the reduced parallelogram, therefore one sees that when the system permits various patterns of this fashion, the sides of the reduced parallelograms will always contain the value $\sqrt{l}$ and $\sqrt{n}$. One would essentially obtain as a result all possible basic parallelograms if one draws lines from o to all adjacent points (always with exemption of the opposite points) and then takes the nearest or the two nearest points in one of the respective nearest parallel lines; and here from the definitively demonstrated (II) this nearest or these nearest points lie closer to o than all points of the subsequent parallel lines, therefore one can see from the condition that the second points are to be taken in the first parallel line. Therefore all possible patterns would be produced if one successively connect o with all point pairs for which the successive minima take place, from which at once follows with consideration from (II) that in general and in the second one of the singular cases obtained then there is only one such pattern, in the first and third exception case however there are respectively two and three patterns of the systems.

In our present reference the precisely obtained singular cases correspond with the suppositions $2m = l < n$, $2m < l = n$, $2m = l = n$.

§4

We have so far only dealt with properties of the geometrical structures which is to be looked at from the theory of forms as the constructive representation of well known theorems and are already indicated in the article cited in the introduction. It is now to solve another problem of another kind, the problem namely when a system of second order is given and a fixed point o of it is examined, to determine the part of the plane every point of which lies nearer to o than to any other points of the system. Here the condition that a point does not lie farther from o than from any other v, therein consists that the point with o on the same side of the perpendicular drawn up in the middle of ov, so we would consequently have o to combine with all remaining points of the system and the convex polygon built from all corresponding perpendiculars to construct. But from these perpendiculars in infinite quantity only a limited number comes into question, while the remaining ones do not meet the polygon determined by it. We abide by all suppositions attended to, so that consequently $op \leqq oq$ in the reduced parallelogram ($poqr$), the angle poq is not obtuse and opq, oqp are acute. This supposes, it is easy to understand, that one has only the six vertices p, q, s, p', q', s' of the four parallelograms meeting at o to take into consideration, and that the perpendiculars corresponding to s and s' and the building diagram in the particular case, when poq is a right angle, only touch, which then the same happens for the perpendiculars corresponding to r and r'. If one draw the straight line pq, os, $p'q'$, os', one obtains the congruent triangles:

$$poq, qos, sop', p'oq', q'os', s'op.$$

If one consider only the points p, q, s, p', q', s', one has to draw a perpendicular in the middle of the straight lines going from o to these points, ie the same construction to make as when one wished t find the middle point of circumscribed circles for the designated triangles. Here no obtuse angle is found in the triangles, therefore each two successive perpendiculars not outside the corresponding triangle intersect. One obtains therefore the hexagon $\alpha\beta\gamma\alpha'\beta',\gamma'$ with the centre o and equal opposite angles and sides as the space, inside of which every point is less far apart from o than from one of the points p,q,s,p',q',s', and one is easily convinced that, with exception of r and r', the perpendiculars corresponding to the remaining points do not meet our hexagon. This requires, as a result of symmetry, only for the points in and above the line pop' to be established. For the former ones it is clear; for the latter ones it would hence appear that their distance from o is larger than the diameter of the circle traced around the hexagon. If one designate the square of its radius ρ, then:

$$4\rho\Delta = ln(l-2m+n),$$

from which as a result of $2m \leqq l, 2m \leqq n, \Delta \geqq \frac{3}{4}ln$, it follows:

$$4\rho \leqq \frac{4}{3}(l-2m+n) \leqq \frac{8}{3}n.$$

Here now for the points of the second and the subsequent parallel lines, as already remarked, the square of their distance from o amounts to at least $3n$, therefore there still remain simply the points in $tsqr$ apart from s,q,r to examine. From all of these none is closer to o than t, for which the square of the distance is equal to $4l-4m+n$, and that this is larger than 4ρ, one immediately sees when one multiplies with Δ and then looks at the inequalities $2m \leqq l \leqq n$. As for the point r, one is also convinced by the same manner that the square of its distance from o is equal to $l+2m+n > 4\rho$, the only case excluded, where $m=0$, in which the corresponding perpendicular touches. It is thus demonstrated that every point in the inside of the hexagon $\alpha\beta\gamma\alpha'\beta'\gamma'$, and only one such hexagon, lies closer to the point o than any other of the system. On any side the distance from o would be equal to the distance from a second point, which for example for $\alpha\beta$ is the point q, and every vertex of the diagram is of the same distance from o and another two other points of the the system. The latter statement undergoes a modification only in the special cases when the angle poq is a right angle; thereupon β and γ as well as β' and γ' coincide, and the hexagon turns into a rectangle, of which the corner from o and another three other points of the system are equally far apart.

It goes without saying that one will always obtain the same hexagon whose reduced parallelogram one also lay as foundation of the construction in the singular cases, where more than one exists, just as also that the hexagon or quadrangle corresponding to all the points of the system are congruent and cover the whole plane of it.

We notice further that, as one is easily convinced, the expression:

$$\rho = \frac{ln(l-2m+n)}{4(ln-m^2)}$$

decreases when one therein, assuming l and n constant, allows m to grow from zero up to its limit $\frac{1}{2}l$, so that consequently:

$$\rho \leqq \frac{1}{4}(l+n) \leqq \frac{1}{2}n. \tag{1}$$

Also in addition the following inequality takes place:

$$2\Delta(n-\rho) \geqq ln^2, \tag{2}$$

the soundness of which is immediately evident when one multiplies with 2, moves everything to one side and then applies $\Delta = ln - m^2$, $4\Delta\rho = ln(l-2m+n)$, by the mean of which it changes into $ln(l-2m) + 2mn(n-l) \geqq 0$.

§5

We come now to our true topic and have to prove that every system of third order can be arranged according to a parallelepiped whose faces are reduced parallelograms and whose edges, from which every four are equal to one another, do not exceed their diagonals.

After one has fixed an arbitrary point (0) of the systems, one would choose in pairs of opposite points for which the distance from (0) is a minimum, or when the minimum of the distance exists for several pairs, would arbitrarily choose a point (1) in one of these pairs. From all points outside the straight line (01) one would again choose one of the two nearest (2), through which again the selection under several pairs, for which the same shortest distance takes place, can be arbitrarily made. Here in the whole system, with exception of the points in (01), no point lies closer to (0) than to (2), so the same is valid also for the plane (102), and (102) is a reduced parallelogram for the system which contains this plane (§3, III). One now takes in one of the two nearest parallel planes the point which is closest to (0) or, when the minimum occurs for more than one, one of the nearest ones and connect (0) with the chosen point (3), therefore the parallelepiped would with the edges (01), (02), (03), as is easy to see, suffice the requirement. Next from the construction it follows: (01) $\leqq$ (02) $\leqq$ (03). Here for the bases of the parallelepiped (we would always indicate as such each face opposite to one another in which are found edges two of which do not exceed the third one in size, and the term side faces apply to the four remaining ones) it is already proven that they are reduced, therefore we have in virtue of the precisely noted doubled inequality only to point out further that the four diagonals of the side faces, just as the four diagonals of the body, are not smaller than (03). Now the eight diagonals mentioned above will agree, as one immediately sees, in size with the eight connecting lines which could be drawn from (0) to the eight points lying around (3) in the plane of the higher bases if we indicate this way for convenience the eight vertices of the four parallelograms meeting at (3). The fact that from the afore-mentioned connection lines none is smaller than (03) follows from the condition by which (3) is being chosen.

After we have convinced ourselves that a system of third order can always be partitioned by a reduced parallelepiped, we now have to establish the relation between such parallelepiped and the system and particularly to compare the distance of the point of systems from (0) with one another. We set $(01) = \sqrt{a}$, $(02) = \sqrt{b}$, $(03) = \sqrt{c}$ and always hold fast the assumption $a \leqq b \leqq c$.

1. In the plane of the base the conditions discused above (§3, II) occurs, so that consequently the successive minima of the distance are always $\sqrt{a}$, $\sqrt{b}$ in size, whereby then in the singular cases mentioned there there is an arbitrariness in the choice of the points.

2. We now look at the points outside the plane of the base underneath namely first of all the one in the plane of the base above. Here from the assumption that our parallelepiped is a reduced one, the line (03) is not longer than one of the straight lines drawn from (0) to the eight points lying around (3), so as a result the foot of the perpendicular dropped from (0) on to the plane of the base above would not be farther apart from (3) than from one of the eight points mentioned. This foot point therefore does not fall outside the hexagon or quadrangle constructed to belong to (3) in the last paragraph. Of those eight points can exceptionally, when the foot point falls on one side, one, or it could, when the foot point meets with a vertex, two (three, when the polygon becomes a rectangle,) lie equally close to the foot point as the point (3), while all remaining points of the plane are further apart from that foot point. It follows from this that the shortest distance (amounting to $\sqrt{c}$) from (0) to the a point in the base above is valid in general only for the point (3), but can exceptionally take place for one, two or even three other points.

3. For the consideration of the following parallel planes we have a boundary for the square h of the perpendicular already mentioned to determine. Here the foot point of the perpendicular does not fall outside the hexagon which belongs to (3), therefore, when ρ denotes the square of the radius of the circumscribed circle:

$$h \geqq c - \rho.$$

Now also from §4: $\rho \leqq \frac{1}{2}b \leqq \frac{1}{2}c$, consequently $h \geqq \frac{1}{2}c$. Here therefore the second parallel plane is at least $\sqrt{2c}$ away, therefore there is over the higher base only points, the distance from (0) of which is greater than $\sqrt{c}$.

If one summarises what has been said, one will see that the minimum of the distance for the entire system has the value $\sqrt{a}$, that, after a point is chosen at this distance, the minimum in the still remaining directions amounts to $\sqrt{b}$, and that finally after the second point is also fixed, for all points outside the plane, which is determined through (0) and the first two points, the smallest distance from (0) is reduced to $\sqrt{c}$. If the successive minima $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ are also always completely determined in quatity, the same minimum in local relation will not be true without several exceptions which are easy to specify. If for example $a \leqq b$, $b < c$, the first two points are to be chosen in the lower base, whereby the singular cases mentioned in §3, II could occur, while the third point lies in the higher base, has a fixed position there in general, in singular cases however can occupy two, three or four different places. One ever so easily overlook that varieties in the other two cases, where $a < b = c$ or $a = b = c$, could happen.

Here from the assumption of a reduced parallelepiped with the edges $\sqrt{a} \leqq \sqrt{b} \leqq \sqrt{c}$ the length of these edges have yielded themselves as the successive minima of the system, thus it immediately follows that when several reduced parallelepiped exist from which the system can be arranged, these all become in agreement with one another with regard to the lengths of their edges, and it could also be easily pointed out that three of the lines directed from (0) to points of the systems of the lengths $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ when they only do not lie in the plane, are always the edges of a reduced parallelepiped. It requires therefore only the easy consideration already applied in a similar case (§3, III). Here after this the entire reduced parallelepipeds would be obtained when one construct the successive minima of every possible types, therefore it becomes evident that when this can happen in only one way (to which we also consider the case where, with the equation of two of the quantities $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ or with the equation of all three, the three lines are locally completely determined and only an exchange between two or all three can occur) that spatial system would allow only one pattern from a reduced parallelepiped. In all other cases there are several such patterns, the parallelepipeds of which form the basis, which could be either all different from one another or only different in part or even could be all congruent to one another. (Similarly in the two singular cases of a system of second order mentioned above the reduced parallelograms underlying the two or three various patterns were congruent to one another.)

To the determination of the question whether a system of third order permits only one or more than one pattern from a reduced parallelepiped, it would consequently only need the knowledge of a single pattern of the system, and the first case would always and exclusively take place when the reduced parallelepiped given through this pattern is of such a property that all lines which can not be exceeded by others actually exceed this parallelepiped, that is when all diagonals of the faces are larger than their sides and all diagonals of the parallelepiped are similarly larger than the edges of the bodies.

§6

As we now apply the results of the last paragraph to the ternary form, shall the uniformity be assumed because of and in order to avoid pointless differentiation, that one has given every ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy \tag{1}$$

through transposition or change of sign of indeterminate elements, as a result of which the form does not belong to the same class, a single form, that firstly $a \leq b \leq c$ holds, that secondly under the coefficients a', b', c', when not the case that all of them are nonzero and are negative, none has the negative sign, and thirdly, when $b = c$ holds, c' apart from the sign not greater than b', when $a = b$ holds, b' not greater than a', and lastly when $a = b = c$ holds, neither c' greater than b' nor b' greater than a' holds. As is easy to see these condition can only be satisfied in one way and their introduction gives the advantage that, as already without these conditions every ternary form corresponds with a completely determined parallelepiped, now to every parallelepiped also belongs an analytical expression the coefficients of which are also completely determined with regard to their sequence and their signs. This assumed, we mention the form (1) in which

$a \leq b \leq c$ also holds, a reduced one, when it corresponds with a reduced parallelepiped. There the diagonal of the area must not be smaller than the sides themselves, so one has:

$$a \pm 2c' + b \geq b, \; a \pm 2b' + c \geq c, \; b \pm 2a' + c \geq c.$$

One sets $\sigma = -1$, where a', b', c' are all three negative, otherwise $\sigma = 1$, so these conditions are synonymous with:

$$a \geq 2c'\sigma, \; a \geq 2b'\sigma, \; b \geq 2a'\sigma, \tag{2}$$

and only, when the equal sign holds, would one of the diagonals in the corresponding parallelogram be equal to a side. The conditions with regard to the diagonals of the parallelepiped result in:

$$a + b + c + 2a'\epsilon + 2b'\delta + 2c'\delta\epsilon \geq c \quad (\delta = \pm 1, \epsilon = \pm 1),$$

where the signs in $\delta = \pm$, $\epsilon = \pm 1$ are arbitrary. One look next at the case where none of the coefficients a', b', c' is negative, and take into account the four sign combinations, as well as that, when a and b are equal to one another then $b' < a'$, so one immediately sees that our inequality is by itself capable of always meet the condition contained, and that the limiting case of the equation in which the diagonals of the edge $\sqrt{c}$ become equal, only once and only then can it occur, when one of the quantities b', c' is equals to zero and when at the same time of the conditions (2) the two quantities b', c' of which are relating to one another, as well as the one which contains a', satisfies the limiting case of the equation. a', b', c' are negative, then our inequality is always fulfilled that the limiting case can not take place, except when $\delta = \epsilon = 1$, so that the consequently the new condition is established:

$$a + b + 2a' + 2b' + 2c' \geq 0, \tag{3}$$

where again the lower sign relates to the equality between a diagonal and the edge $\sqrt{c}$.

The condition just developed (2) and, when a', b', c' are negative, (3) above is are therefore fulfilled, that the inequality takes place in none of the inequalities of the limiting case, therefore in the class to which the form belongs it would not give a second one of these various ones with or without equality signs in the definition condition, here according to at the end of the last paragraph notice that the corresponding system of points can only be partitioned from a reduced parallelepiped. The matter stands differently when the upper signs do not take place in all conditions; there could then occur in the same class several reduced forms that can be derived from a given one. It is sufficient to demonstrate this for a main case. We choose for this the case where $b < c$.

Next one assumes $a > 2c'\sigma$, therefore the direction of the edge $\sqrt{c}$ can only be altered when there are namely in the plane of higher base still one or more points, the distance from the vertex of which amounts to $\sqrt{c}$. When ξ, η are 1 the one such points corresponding values of the element so would, when the third edge depends on it, all the coefficients except a', b' remain unchanged, these respectively change into $a' + c'\xi + b\eta$, $b' + a\xi + c'\eta$ as one is easily and almost convinced without calculation. Now from the specification made earlier are the values of ξ, η which meet the condition, when $a = 2b'\sigma$:

$$\xi = -\sigma, \quad \eta = 0;$$

when $b = 2a'\sigma$:

$$\xi = 0, \quad \eta = -\sigma;$$

when simultaneously $a = 2b', b = 2a', c' = 0$:

$$\xi = -1, \quad \eta = -1;$$

and when a', b', c' are negative and the equation $a + b + 2a' + 2b' + 2c' = 0$ complied with:

$$\xi = 1, \quad \eta = 1.$$

Corresponding to these four assumptions one has consequently transformed a', b' into:

$$a' - c'\sigma, \; b' - a\sigma (= -b'); \; -a', \; b' - c'\sigma; \; -a', \; -b'; \; a' + b + c', \; a + b' + c'.$$

From the third case and generally from the assumption $c' = 0$ one can foresee, here this represents a new form which afterwards one has undertaken in the same one the change of sign stated in the beginning of this paragraph, apparently with the form from which one has derived, become identical. In each of the three remaining assumptions one obtains from application of the necessary sign change a new reduced form belonging to the same class (provided that it does not coincide with the original one), and one obtains two such forms when two of our assumptions hold at the same time. With this then the specification of the form is brought to an end, here apparently the simultaneity of all three assumptions can not take place. If, always under the assumption $b < c$, $a = 2c'\sigma$, one would have, provided that $a < b$, the two edges rotated in the base, for $a = b$ the first edge can also pass into the original position of the two first edges and this new position or both of these new positions of the first two edges must be associated with all the directions of the third one, the original one not excluded.

One can thereby easily remove the inconvenience that in singular cases several reduced forms can be associated in the same class, and thereby take away the exception that for such singular cases in general definition one still include the known secondary conditions which can be proved when one for instance set up the demand that the last coefficient c', provided that it is not fully fixed, maintains the smallest numerical value of which it is capable in the reduced forms of the class, and then likewise with regard to b'. For this to notify an example, we will examine it under the singular cases dealt with earlier where $b < c$, from the three conditions (2) none with the lower signs holds, however the three negative values a', b', c' satisfy the equation:

$$a + b + 2a' + 2b' + 2c' = 0.$$

From the previous observations c' is fixed, and there exists for this case only two reduced forms. a' and b' are the values of the forth and the fifth coefficients in one of them, therefore they are in the other $a'+b+c'$, $a+b'+c'$, or, here the last value are apparently positive, c' is negative and consequently, in order for the sign specification to be sufficient, z to transform into $-z$, rather $-(a'+b+c')$, $-(a+b'+c')$. As it is quite natural, these values suffice when one substitute them for a', b', again the equation:

$$a+b+2a'+2b'+2c'=0,$$

and from them come the values a', b' in the same manner, as they themselves are originated from a', b'. Here according to this the fifth coefficient admits only the two negative values b' and $-(a+b'+c')$, their sum equals $-a-c'$, therefore one see that when one further add to the definition conditions:

$$-b' \leqq \frac{1}{2}(a+c'),$$

the class would contain only one reduced form.

While we conclude the essay, we will still from our principles derive a beautiful theorem, found by *Seeber* through induction and demonstrated by *Gauss* in the announcement already often quoted. From this theorem in a reduced form the production of the first three coefficients is not larger than the doubled absolute value of the determinant.

Here the absolute value of the determinant is equal to the square of the volume of the parallelepiped corresponding to the form, thus consequently, from the expression employed in §5, 3 the inequality to be proved:

$$abc \leqq 2\Delta h,$$

where Δ represents the square of the base. One sets:

$$c=b+t,$$

where consequently t is not negative, draws off from the inequality obtained in §5, 3:

$$h \geqq c-\rho=b-\rho+t,$$

from which one has multiplied it with 2Δ, the equation:

$$abc=ab^2+abt,$$

thus one obtains:

$$2\Delta h-abc \geqq 2\Delta(b-\rho)-ab^2+(2\Delta-ab)t.$$

Here now from the inequality at the end of §4, $2\Delta(b-\rho)-ab^2$ is not negative and $2\Delta-ab \geqq \frac{1}{2}ab$ is positive, therefore the truth of the theorem becomes evident.

§ F.2 G. F. Voronoi, 1908 (I)

New applications of continuous parameters to the theory of the quadratic form
First Memoir
On some properties of the perfect positive quadratic forms
by Mr. Georges Voronoï in Warsaw

[*Journal für die reine und angewandte Mathematik*, V. 133, p. 97–178, 1908]
[translated by K N Tiyapan]

Introduction

Hermite had introduced in the theory of numbers a new and fruitful principle, namely: being given a set (x) of systems $(x_1, x_2, \dots, x_n)$ for all the values of $x_1, x_2, \dots, x_n$, one associates with the set (x) a set (R) composed of the domains in a manner such that by studying the set (R) one studies at the same time the set (x) .

Hermite has shown † numerous applications of the new principle to the generalisation of continuous fractions, to the study of algebraic units, etc.

The ideas of Hermite have been developed in the works of Mr.'s Zolotareff, Charve, Selling, Minkowski. ‡

I intend to publish a series of Mémoires in which I shall show new applications of the principle of Hermite to the various problems of the arithmetic theory of definite and indefinite quadratic forms.

In this Mémoire, I study the properties of the minimum of positive quadratic forms and of their various representations by systems of integers.

Hermite has discovered an important property of the minimum M of positive quadratic forms $\sum a_{ij}x_ix_j$ in n variables and of the determinant D, namely:

$$M \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D},$$

and he has demonstrated numerous applications of this formula.

In a letter to Jacobi, Hermite has said §:

"That which precedes sufficiently indicates an infinity of other analogous consequences which, all, will depend on the difficult study of an exact limit of the minimum of any definite form. Thereupon I then form only one conjecture. My first studies in the case of a form in n variables of the determinant D have given me the limit $\left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D}$, I am inclined to presume, but without being able to demonstrate that the numerical coefficient $\left(\frac{4}{3}\right)^{\frac{n-1}{2}}$ has to be replaced by $\frac{2}{\sqrt[n]{n+1}}$"

Mr.'s Korkine and Zolotareff has under taken the study of the exact limit of the minimum of positive quadratic forms of the same determinant.

By indicating with $M(a_{ij})$ the minimum and with D (a_{ij}) the determinant of the form $\sum a_{ij}x_ix_j$, one will have the minimum

$$\mathcal{M}(a_{ij}) = \frac{M(a_{ij})}{\sqrt[n]{D(a_{ij})}}$$

of a positive quadratic form with determinant 1.

By virtue of the theorem of Hermite the function $\mathcal{M}(a_{ij})$ verifies the inequality ¶

$$\mathcal{M}(a_{ij}) \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}},$$

therefore it is bounded within the set (f) of all the positive quadratic forms of real coefficients.

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† *Hermite.* Extraits de lettres de M. Ch. Hermite à M. Jacobi sur differents objets de la théorie des nombres. [Excerpts from letters of Mr. Ch. Hermite to Mr. Jacobi on various subjects in the theory of numbers] (This Journal V. 40, p. 261)

Hermite. Sur l'Introduction des variables continues dans la théorie des nombres. [On the introduction of the continuous variables in the theory of numbers] (This Journal V. 41, p. 191)

Hermite. Sur la théorie des formes quadratiques. [On the theory of quadratic forms] (This Journal V. 47, p 313)

‡ *Zolotareff.* On an indeterminate equation of the third degree (Petersbourg, 1869, in Russian.)

Zolotareff. Theory of complex integers with applications to the integral calculus. (Petersbourg, 1874, in Russian.)

Charve. De la réduction des formes quadratiques ternaires positives et de leur application aux irrationelles de troisième degré. [Of the reduction of positive ternary quadratic forms and of their application to the irrationals of third detree] (Suppl. to V. IX of Annales Scientifiques de l'Ecole Normale Supérieure, 1880)

Selling. Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

Minkowski. Geometrie der Zahlen. [Geometry of numbers] (Leipzig, 1896)

§ This Journal. V 40, p. 296

¶ Mr. Minkowski has demonstrated an upper limit of the function $\mathcal{M}(a_{ij})$

$$\mathcal{M}(a_{ij}) \leq n$$

much simpler than that from Hermite.

(*Minkowski.* Über die positiven quadratischen Formen und über kettenbruchähnliche Algorithmen. [On the positive quadratic forms and on continued fraction algorithm] This Journal V. 107, p. 291)

Mr.'s Korkine and Zolotareff have demonstrated † that the function $\mathcal{M}(a_{ij})$ possesses many maxima in the set (f) which correspond to the various classes of equivalent positive quadratic forms.

The limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite in the letter to Jacobi (source cited) is only a maximum value of the function $\mathcal{M}(a_{ij})$.

The binary and ternary positive quadratic forms possess a single maximum which is therefore, in this case, the exact limit of values of the function $\mathcal{M}(a_{ij})$.

Reckoning from the number of variables $n \geq 4$, one meets many maxima of the function $\mathcal{M}(a_{ij})$.

Mr.'s Korkine and Zolotareff have found many values of various maxima of the function $\mathcal{M}(a_{ij})$ which exceed the limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite, but do not exceed the limit 2.

The study of the exact limit of the minimum of positive quadratic forms of the equal determinant comes down, after Mr.'s Korkine and Zolotareff, to the study of all the various classes of positive quadratic forms to which correspond the maximum values of the function $\mathcal{M}(a_{ij})$.

The maximum maximorum of values of the function $\mathcal{M}(a_{ij})$ is the largest value of the function $\mathcal{M}(a_{ij})$ which presents a numerical function as $\mu(n)$.

Mr.'s Korkine and Zolotareff have determined the following values of the function $\mu(n)$:

$$\mu(2) = \sqrt{\frac{4}{3}},\ \mu(3) = \sqrt[3]{2},\ \mu(4) = \sqrt[4]{4},\ \mu(5) = \sqrt[5]{8},.$$

They have called extreme the quadratic forms which yield to the function $\mathcal{M}(a_{ij})$ a maximum value.

The extreme quadratic forms enjoy an important property, namely:

I. Any extreme quadratic form is determined by the value of its minimum and by all the representations of the minimum.

Mr.'s Korkine and Zolotareff have determined all the classes of extreme forms in 2, 3, 4 and 5 vertices.

By studying these extreme forms, I have observed that they are all well defined by the property (I). There is only reckoning from positive forms in six variable which I have encountered positive quadratic forms which enjoyed the property (I) and are not of extreme forms.

I call "perfect" any positive quadratic form which enjoys the property (I).

I demonstrate that the set of all the perfect forms in n variables can be divided into classes the number of which is finite.

All extreme form being, by virtue of the property I, a perfect form, it results in that the function $\mu(n)$ presents the maximum of values of the function $\mathcal{M}(a_{ij})$ which correspond to the various classes of perfect forms.

I have established an algorithm for the search of various perfect forms by introducing a definition of contiguous perfect forms.

To that effect, I make correspond to the set (φ) of all the perfect forms in n variables a set (R) of domains in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

The set (R) of domains in $\frac{n(n+1)}{2}$ dimensions presents a partition of the set (f) of all the positive quadratic forms in n variables.

Each domain R possesses in the set (R) a contiguous domain which is well determined by any one face in $\frac{n(n+1)}{2} - 1$ dimensions of the domain R.

I demonstrate that the domain R corresponding to the perfect form $\varphi(x_1, x_2, \ldots, x_n)$ being determined by the linear inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

one will have σ perfect forms defined by the equalities

$$\varphi_k(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_k \Psi_k(x_1, x_2, \ldots, x_n), \qquad (k = 1, 2, \ldots, \sigma) \tag{1}$$

where

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j,$$

provided that the positive parameter ρ_k $(k = 1, 2, \ldots, \sigma)$ presents the smallest value of the function

$$\frac{\phi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)}$$

where $\Psi(x_1, x_2, \ldots, x_n) < 0$ and M is minimum of the form $\phi(x_1, x_2, \ldots, x_n)$.

I call "contiguous to the perfect form $\phi(x_1, x_2, \ldots, x_n)$" the perfect forms (1).

Any substitution in integer coefficients and with determinant ±1 belonging to the group g of substitutions which do not change the form ϕ permute only the forms (1). One can, therefore, divide the forms (1) into classes of equivalent forms with the help of substitutions of the group g. By choosing one form in each class, one will have a system of perfect forms contiguous to the perfect form ϕ which can replace the system (1).

By proceeding in this manner, one can obtain a system complete of representatives of various classes of perfect forms.

The corresponding domains will form complete system of representatives of various classes of the set (R).

I have remarked that a similar system

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{2}$$

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† *Korkine* and *Zolotareff.* Sur les formes quadratiques. [On the quadratic forms] Mathematische Annalen, V. VI, p. 366 and V. XI, p. 242

of domains of the set (R) can serve towards the reduction of positive quadratic forms.

I call reduced any positive quadratic form belonging to one of the domains (2).

It results from this definition:

I. Any positive quadratic form can be transformed into an equivalent reduced form, with the help of a substitution which presents a product of substitutions belonging to a series of substitutions

$$S_1, S_2, \ldots, S_m$$

which depend only on the choice of the system (2).

II. Two reduced forms can be equivalent only provided that the corresponding substitution belonged to a series of substitutions the number of which is finite.

The weak point of the new method of reduction of positive quadratic forms, demonstrated in this Mémoire, consists in that the number of substitutions which transform into itself the domains of the set (R) or their faces is, in general, very large.

The application of the general theory demonstrated in this Mémoire to the numerical examples will be particularly facilitated if one knew how to solve the following problem:

Being given a group G of substitutions which transform into itself a domain R, one would like to partition this domain into equivalent parts the number of which will be equal to the number of substitutions of the group G and on condition that the number of faces in $\frac{n(n+1)}{2} - 1$ dimensions of domains obtained be the smallest possible.

I show in this Mémoire the solution of the problem introduced in two cases: $n = 2$ and $n = 3$.

From the number of variable $n \geq 4$, I do not know any practical solution of the problem posed.

First Part

General theory of perfect positive quadratic forms and domains which correspond to them.

Definition of perfect quadratic forms.

1

Let

$$\phi(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j \tag{1}$$

be any positive quadratic form. By indicating with

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}) \tag{2}$$

the various representations of the minimum M of the form $\sum a_{ij} x_i x_j$, one will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M, \quad (k = 1, 2, \ldots, s) \tag{3}$$

One will not consider in the following the two systems

$$(l_{1k}, l_{2k}, \ldots, l_{nk}) \quad \text{and} \quad (-l_{1k}, -l_{2k}, \ldots, -l_{nk}), \quad (k = 1, 2, \ldots, s)$$

as different and one will arbitrarily choose one of these systems.

On the ground of the supposition made, one will have the inequality

$$\sum a_{ij} x_i x_j > M$$

provided that a system $(x_1, x_2, \ldots, x_n)$ of integer values of variables $x_1, x_2, \ldots, x_n$ did not belong to the series (2), excluding the system $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.

By considering the equalities (3) as the equations which serve to determine $\frac{n(n+1)}{2}$ coefficients of the quadratic form $\sum a_{ij} x_i x_j$, one will have only two cases to examine:

1.) there exist a finite number of solutions of equations (3),

2.) the equations (3) admit only a single system of solutions.

2

Let us examine the first case,

Let us suppose that there exists an infinite number of solutions of equations (3).

One will find in this case an infinite number of values of parameters

$$p_{ij} = p_{ji}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

verifying the equations

$$\sum p_{ij} l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, s) \tag{4}$$

the values $p_{ij} = 0,\ i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$ being excluded.

By indicating

$$\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j,$$

let us consider the set of positive quadratic forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho \Psi(x_1, x_2, \ldots, x_n), \tag{5}$$

the parameter ρ being arbitrary.

For a quadratic form determined by the equality (5) to be positive, it is necessary and sufficient that the corresponding value of the parameter ρ be continuous in a certain interval

$$-R' < \rho < R.$$

It can turn out that $R = +\infty$, in this case the lower limit $-R'$ will be finite. By replacing in the equality (5) the form $\Psi(x_1, x_2, \dots, x_n)$ by the form $-\Psi(x_1, x_2, \dots, x_n)$, that which is permitted by virtue of (4), one will have the interval

$$-R < \rho < R',$$

therefore one can suppose that the upper limit R is finite.

The corresponding quadratic form, determined by the equality

$$f(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n) + R\Psi(x_1, x_2, \dots, x_n),$$

will not be positive, but it will not have negative values either; one concludes that least for a system $(\xi_1, \xi_2, \dots, \xi_n)$ of real values of variables $x_1, x_2, \dots, x_n$ the form $f(x_1, x_2, \dots, x_n)$ attains in its value the smallest which is zero, and it follows that the system $(\xi_1, \xi_2, \dots, \xi_n)$ verifies the equation

$$\frac{\partial f}{\partial \xi_i} = \frac{\partial \varphi}{\partial \xi_i} + R\frac{\partial \Psi}{\partial \xi_i} = 0. \quad (i = 1, 2, \dots, n)$$

By eliminating from these equations $\xi_1, \xi_2, \dots, \xi_n$ one obtains the equation

$$D(R) = \begin{vmatrix} a_{11} + RP_{11}, & a_{12} + RP_{12}, & \dots, & a_{1n} + RP_{1n} \\ a_{21} + RP_{21}, & a_{22} + RP_{22}, & \dots, & a_{2n} + RP_{2n} \\ & \dots & & \\ a_{n1} + RP_{n1}, & a_{n2} + RP_{n2}, & \dots, & a_{nn} + RP_{nn} \end{vmatrix} = 0.$$

The smallest positive root of this equation presents the value of R searched for.

3 Let us examine the set (f) of positive quadratic forms determined by the equality (5) with condition

$$0 < \rho < R. \tag{6}$$

Theorem. To the set (f) belongs a quadratic form $\varphi_1(x_1, x_2, \dots, x_n)$ which is well determined by the following conditions:

1. all the representations of the minimum M of the form $\varphi(x_1, x_2, \dots, x_n)$ are also representations of the minimum M of the form $\varphi_1(x_1, x_2, \dots, x_n)$,

2. the form $\varphi_1(x_1, x_2, \dots, x_n)$ moreover possesses at least another representation of the minimum M.

Let us indicate by $M(\rho)$ the minimum and by $D(\rho)$ the determinant of the quadratic form $f(x_1, x_2, \dots, x_n)$ defined by the equality (5) with condition (6).

By virtue fo the theorem by Hermite, one will have the inequality

$$M(\rho) \leq \mu(n) \sqrt[n]{D(\rho)}. \tag{7}$$

We have demonstrated that $D(R) = 0$, it results in that a value of the parameter ρ can be chosen in the interval (6) such that the inequality

$$\mu(n) \sqrt[n]{D(\rho)} < M$$

holds. One will have, because of (7),

$$M(\rho) < M. \tag{8}$$

Let us indicate by $(l_1, l_2, \dots, l_n)$ a representation of the minimum $M(\rho)$ of the form $f(x_1, x_2, \dots, x_n)$ verifying the inequality (8).

One will have

$$\varphi(l_1, l_2, \dots, l_n) + \rho\Psi(l_1, l_2, \dots, l_n) < M, \tag{9}$$

and as a result

$$\varphi(l_1, l_2, \dots, l_n) > M \text{ and } \Psi(l_1, l_2, \dots, l_n) < 0. \tag{10}$$

This posed, let us find the smallest value of the function

$$\frac{\varphi(x_1, x_2, \dots, x_n) - M}{-\Psi(x_1, x_2, \dots, x_n)} \tag{11}$$

determined with condition

$$\Psi(x_1, x_2, \dots, x_n) \leq 0. \tag{12}$$

To that effect, let us examine the inequality

$$\frac{\varphi(x_1, x_2, \dots, x_n) - M}{-\Psi(x_1, x_2, \dots, x_n)} \leq \frac{\varphi(l_1, l_2, \dots, l_n) - M}{-\Psi(l_1, l_2, \dots, l_n)}.$$

By virtue of (9), (10) and (12), one will have

$$\varphi(x_1, x_2, \dots, x_n) + \rho\Psi(x_1, x_2, \dots, x_n) < M.$$

The quadratic form $\varphi(x_1, x_2, \dots, x_n) + \rho\Psi(x_1, x_2, \dots, x_n)$ being positive, there exists only a limited number of integer values of $x_1, x_2, \dots, x_n$ verifying this inequality. Among these systems are found all the systems which give back to the function (11) the smallest value determined with condition (12).

Let us indicate by

$$\left(l'_1, l'_2, \dots, l'_n\right), \left(l''_1, l''_2, \dots, l''_n\right), \dots, \left(l^{(r)}_1, l^{(r)}_2, \dots, l^{(r)}_n\right)$$

all the representations of the positive minimum ρ_1 of the function (11).

By declaring

$$\varphi_1(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_1 \Psi(x_1, x_2, \ldots, x_n),$$

one obtains the positive quadratic form $\varphi_1(x_1, x_2, \ldots, x_n)$ the minimum M of which is represented by the systems (2) and (13), this is that which one will demonstrate without trouble.

4

With the help of the procedure previously shown, one will determine a series of positive quadratic forms

$$\varphi, \varphi_1, \varphi_2, \cdots \tag{14}$$

which enjoy the following property: by indicating with s_k the number of representations of the minimum of the form $\varphi_k (k = 1, 2, \ldots)$, one will have the inequalities

$$s < s_1 < s_2 < \cdots \tag{15}$$

A similar series of positive quadratic forms of n variables can not be extended indefinitely, this is that which we will demonstrate with the help of the following lemma.

Lemma. The number of various representations of the minimum of a positive quadratic form in n variables does not exceed $2^n - 1$.

Let us indicate by $(l_1, l_2, \ldots, l_n)$ and $(l'_1, l'_2, \ldots, l'_n)$ any two representations of the minimum M of the positive quadratic form $\sum a_{ij} x_i x_j$.

Let us suppose that by declaring

$$l'_i = l_i + 2t_i, \quad (i = 1, 2, \ldots, n) \tag{16}$$

the number $t_1, t_2, \ldots, t_n$ would be integer.

As

$$\sum a_{ij} l'_i l'_j = M \quad \text{and} \quad \sum a_{ij} l_i l_j = M,$$

by virtue of (16), it becomes

$$\sum a_{ij} l_i t_j + \sum a_{ij} t_i t_j = 0$$

One will present this equality under the form

$$\sum a_{ij}(l_i + t_i)(l_j + t_j) + \sum a_{ij} t_i t_j = \sum a_{ij} l_i l_j. \tag{17}$$

By noticing that

$$\sum a_{ij} t_i t_j \geq \sum a_{ij} l_i l_j,$$

one finds, by virtue of (17),

$$\sum a_{ij}(l_i + t_i)(l_j + t_j) \leq 0,$$

therefore it is necessary that

$$\sum a_{ij}(l_i + t_i)(l_j + t_j) = 0,$$

and consequently

$$l_i + t_i = 0. \quad (i = 1, 2, \ldots, n)$$

Because of (16), one obtains

$$l'_i = -l_i. \quad (i = 1, 2, \ldots, n)$$

This posed, let us divide the set (X) of all the systems $(x_1, x_2, \ldots, x_n)$ of integer values of $x_1, x_2, \ldots, x_n$ into 2^n classes, with regard to the modulo 2.

We have demonstrated that two different representations of the minimum M of the form $\sum a_{ij} x_i x_j$ will not belong to the same class; neither will any representation of the minimum M belong to the class made up of systems $(x_1, x_2, \ldots, x_n)$ satisfying the conditon

$$x_i \equiv 0 (\mathrm{mod} 2), \quad (i = 1, 2, \ldots, n)$$

therefore the number of various representations of the minimum of a positive quadratic form can not be greater than $2^n - 1$.

5

We have demonstrated that the series (14) of positive quadratic forms satisfying the condition (15) can not be extended indefinitely, therefore the series (14) will be terminated by a form φ_k which enjoys the following property: the form φ_k is determined by the representations of its minimum.

Definition. One will call perfect any positive quadratic form which is determined by the representations of its minimum.

Let us suppose that the form (1) be perfect, one will have in this case only a single system of solutions of equations (3).

On the ground of the supposition made, the equations

$$\sum p_{ij} l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, s)$$

admit only a single system of solutions

$$p_{ij} = p_{ji} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By effecting the solution of equations (3), one obtains the equalities

$$a_{ij} = \alpha_{ij} M, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where the coefficients α_{ij} are rational.

It results in that the perfect form $\frac{\varphi}{M}$ is of rational coefficients. In the following one will not consider as different the perfect forms of proportional coefficients.

Fundamental properties of perfect quadratic forms.

6 Let

$$\varphi(x_1, x_2, \ldots, n) = \sum a_{ij} x_i x_j$$

be a perfect quadratic form. Let us suppose that all the different representations of the minimum of the perfect form φ make up the series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \tag{1}$$

By choosing any n systems in this series, let us examine the determinant

$$\begin{vmatrix} l_{11}, & l_{12}, & \ldots, l_{1n} \\ l_{21}, & l_{22}, & \ldots, l_{2n} \\ & \ldots & \\ l_{n1}, & l_{n2}, & \ldots, l_{nn} \end{vmatrix} = \pm\omega. \tag{2}$$

All the determinants that one can form this way can not cancel each other out. By supposing the contrary, one will have s equations of the form

$$l_{ik} = \sum_{r=1}^{n-1} l_{ir} u_r^{(k)}, \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, s) \tag{3}$$

One will choose a system of $\frac{n(n+1)}{2}$ parameters $p_{ij} = p_{ji}$ verifying $\frac{n(n+1)}{2}$ equations

$$\sum p_{ij} l_{ir} l_{jt} = 0, \quad (r = 1, 2, \ldots, n-1; t = 1, 2, \ldots, n-1)$$

and by virtue of (3), one will have

$$\sum p_{ij} l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, s)$$

which is impossible.

The numerical value ω of the determinant (2) can not exceed a fixed limit. To demonstrate this, let us effect a transformation of the perfect form φ with the help of a substitution

$$x_i = \sum_{r=1}^{n} l_{ir} x'_r; \quad (i = 1, 2, \ldots, n) \tag{4}$$

one will obtain a form

$$\varphi'(x'_1, x'_2, \ldots, x'_n) = \sum a'_{ij} x'_i x'_j,$$

where

$$a'_{ii} = M. \ (i = 1, 2, \ldots, n) \tag{5}$$

By indicating with D' the determinant of the form φ', one will have the inequality

$$a'_{11} a'_{22} \cdots a'_{nn} \geq D',$$

by virtue of the known property of positive quadratic forms.

Considering (5), one obtains

$$M^n \geq D'. \tag{6}$$

By indicating with D the determinant of the form φ, one will have, because of (2) and (4),

$$D' = D\omega^2,$$

therefore the inequality (6) reduces to the one here:

$$D\omega^2 \leq M^n.$$

By virtue of the theorem by Hermite, one has the inequality

$$M \leq \mu(n) \sqrt[n]{D};$$

it follows that

$$\omega \leq [\mu(n)]^{\frac{n}{2}}. ^{\dagger} \tag{7}$$

7

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† See the Mémoire of Mr.'s Korkine and Zolotareff *sur les formes quadratiques positives.* (Mathematische Annalen V. XI, p. 256)

Any perfect form will obviously be transformed into a form, also perfect, with the help of all linear substitution of integer coefficients and of determinant ±1.

One concludes this that there exists a finite a finite number of equivalent perfect forms.

The set (φ) of all the perfect forms in n variables can be divided into different classes provided that each class be made up of all the equivalent perfect forms.

Theorem. The number of different classes of perfect forms in n variables is finite.

Let us indicate by

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n \quad (k = 1, 2, \ldots, s)$$

s linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s \tag{8}$$

which correspond to the systems (1) of representations of the minimum of the form Ψ.

One establishes this way a uniform correspondence between a perfect form φ and the system (8) of linear forms.

Let us suppose that one had transformed the perfected form φ with the help of a substitution S by integer coefficients and with determinant ±1, one will obtain an equivalent perfect form φ'. Let us indicate by

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{9}$$

the corresponding system of linear forms.

One will easily demonstrate that the substitution T, adjointed to the substitution S §, will transform the system (8) into a system (9).

One concludes that a certain reduction of perfect forms can be effected with the help of the reduction of corresponding systems of linear forms.

The reduction of the system (8) comes down, by virtue of (7), to the reduction of any n linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_n \tag{10}$$

belonging to the system (8) and with determinant $\pm\omega$ which does not cancel each other out.

One will determine with the help of the known method a substitution T which will transform the linear forms (10) of integer coefficients into linear forms,

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_n \tag{11}$$

satisfying to the following conditions

$$\begin{cases} \lambda'_k = p_{k,k}x' + p_{k+1,k}x'_{k+1} + \ldots + p_{n,k}x'_n, & (k = 1, 2, \ldots, n) \\ p_{11}p_{22}\cdots p_{nn} = \omega \text{ and } p_{kk} > 0, & (k = 1, 2, \ldots, n) \\ 0 \le p_{k+i,k} < p_{kk}. & (i = 1, 2, \ldots, n-k; k = 1, 2, \ldots, n) \end{cases}$$

The coefficients of forms (11) being integers, as a result they do not exceed fixed limits.

The substitution T will transform the system (8) into a system

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{12}$$

of linear forms. By examining successively the determinants of forms

$$(\lambda'_k, \lambda'_2, \ldots, \lambda'_n), (\lambda'_1, \lambda'_k, \ldots, \lambda'_n), \ldots, (\lambda'_1, \lambda'_2, \ldots, \lambda'_k), \ (k = n+1, n+2, \ldots, s)$$

one will demonstrate that the numerical values of coefficients of all the linear forms (12) do not exceed fixed limits.

The number of similar systems of linear forms in integer coefficients being limited, it results in that the number of different classes of perfect forms is also limited.

8 *On the domains determined with the help of linear inequalities*

We have seen in Number 7 that the study of perfect forms can be brought back to the study of certain systems of linear forms.

One will acquire a new basis to these studies by making correspond to each perfect quadratic form in n variables a domain in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

One will address first the general problem by studying the properties of domains determined with the help of linear inequalities. ‡

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§ The substitution S being defined by the equalities

$$x_i = \sum_{k=1}^{n} \alpha_{ik}x'_k, \quad (i = 1, 2, \ldots, n)$$

one calls "substitution adjointed to S" the substitution T which is determined by the equalities

$$\sum_{k=1}^{n} \alpha_{ik}x_k = x'_i. \quad (i = 1, 2, \ldots, n)$$

‡ See: *Minkowski.* Geometrie der Zahlen [Geometry of the numbers], No. 19, p. 39.

Let us consider a system of linear inequalities

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

in any real coefficients.

One will call point (x) any system $(x_1, x_2, \ldots, x_m)$ of real values of variables $x_1, x_2, \ldots, x_m$ and one will indicate

$$y_k(x) = p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m. \quad (k = 1, 2, \ldots, \sigma)$$

One will call "domain" the set R of points verifying the inequalities

$$y_k(x) \geq 0. \ (k = 1, 2, \ldots, \sigma) \tag{1}$$

Let us suppose that to the domain R belonged to points verifying the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

one will call such points interior to the domain R, and the domain R will be said to be of m dimensions.

It can be the case that the domain R does not possess interior points. One will demonstrate in this case all the points belonging to the domain R verify at least one equation

$$y_k(x) = 0,$$

the indice h being a value $1, 2, \ldots, \sigma$.

It is important to have a criteria with the help of which one could recognise whether a domain determined by the help of inequalities (1) will be in m dimensions or not.

Fundamental principle. For a domain determined with the help of inequalities (1) to be of m dimensions, it is necessary and sufficient that the equation

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \tag{Ξ}$$

did not reduce into an identity so long as the parameters $\rho_2, \rho_2, \ldots, \rho_\sigma$ are positive or zero, the values $\rho_1 = 0, \rho_2 = 0, \ldots, \rho_\sigma = 0$ being excluded.

The principle introduced, considered from a certain point of view, is evident, but one arrive at the rigorous demonstration of this principle only with the help of the in depth study of domains determined with the help of linear inequalities.

For more simplicity, one will examine in that which follows only domains satisfying the following conditions: the equations

$$y_k(x) = 0 \ \ (k = 1, 2, \ldots, \sigma) \tag{2}$$

can not be verified by any point, the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.

It is easy to demonstrate that the general case will always come down to the case examined.

9 *Definition. One will call edge of the domain R determined with the help of inequalities (1) the set of points belonging to the domain R and verifying the equations*

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, r \ \text{ where } \ r < \sigma)$$

provided that these equations defined the values of $x_1, x_2, \ldots, x_m$ to an immediate common factor.

By indicating with $(\xi_1, \xi_2, \ldots, \xi_m)$ a point of the edge considered, one will determine all the points of the edge with the help of equalities

$$x_i = \rho\xi_i, \quad (i = 1, 2, \ldots, n)$$

ρ being an arbitrary positive parameter.

This results in that each edge of the domain R is well determined by any point belonging to it.

Let us suppose that the domain R possesses s edges characterised by the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}). \quad (k = 1, 2, \ldots, s)$$

By declaring

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik}, \ \ (i = 1, 2, \ldots, m) \tag{3}$$

where

$$\rho_k \geq 0, \quad (k = 1, 2, \ldots, s) \tag{4}$$

one obtains a point (x) belonging to the domain R, the positive or zero parameters $\rho_1, \rho_2, \ldots, \rho_s$ being arbitrary.

10 *Fundamental theorem. Let us suppose that the inequalities (1) which define the domain R satisfy the condition (Ξ).*

The domain R will be of m dimensions and each point belonging to it will be determined by the equalities (3) with condition (4).

The theorem introduced is well known in the case $m = 2$ and $m = 3$.

We will demonstrate that by supposing that the theorem be true in the case of $m - 1$ variables, the theorem will again be true in the case of m variables.

Let us examine first the various inequalities of the system (1). It can be the case that many among them could be put under the form

$$y_h(x) = \sum_{k=1}^{s} \rho_k^{(h)} y_k(x) \quad \text{where} \quad \rho_k^{(h)} \geq 0. \quad (k = 1, 2, \ldots, s; \rho_h^{(h)} = 0)$$

One will call such inequalities dependent and one will exclude them from the system (1).

Let us suppose that the system (1) contained only independent inequalities.

Their number ρ, on the ground of the supposition (2) made, will not be less than m.

This posed, let us examine a set P_h of points belonging to the domain R and verifying an equation

$$y_h(x) = 0, \tag{5}$$

the indice h having a value $1, 2, \ldots, \sigma$.

One will call "face of the domain R" the domain P_h.

On the ground of the supposition made, the face P_h will be in $m - 1$ dimensions.

To demonstrate this, let us make correspond to any point (x) verifying the equation (5) a point (u) in $m - 1$ coordinates $(u_1, u_2, \ldots, u_{m-1})$ by declaring

$$x_i = \sum_{j=1}^{m-1} \alpha_{ij} u_j. \quad (i = 1, 2, \ldots, m)$$

The system of inequalities (1) will be transformed into a system

$$\eta_k(u) \geq 0 \quad (k = 1, 2, \ldots, \sigma; k \neq h) \tag{7}$$

of inequalities in $m - 1$ variables $u_1, u_2, \ldots, u_{m-1}$.

Let us suppose that one knew how to reduce the equation

$$\sum_{k=1}^{\sigma} \rho_k \eta_k(u) = 0 \quad \text{where} \quad \rho_h = 0 \quad \text{and} \quad \rho_k \geq 0 \quad (k = 1, 2, \ldots, \sigma) \tag{8}$$

into an identity. By virtue of (6), one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = \rho y_h(x) \quad \text{where} \quad \rho_h = 0.$$

One can not suppose that $\rho > 0$, since otherwise the inequality

$$y_h(x) \geq 0$$

would be dependent and on the ground of the supposition made would not belong to the system (1).

By supposing that $\rho \leq 0$, one will admit $\rho_h = -\rho$ and one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \quad \text{where} \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

which is contrary to the hypothesis.

We have supposed that the theorem introduced be true in the case of $m - 1$ variables. As the equation (8) can not be reduced into an identity, one concludes that the system of inequalities (7) defines a domain $\mathcal{B}_h$ in $m - 1$ dimensions. Moreover, by indicating with

$$(u_{11}, u_{21}, \ldots, u_{m-1,1}), (u_{12}, u_{22}, \ldots, u_{m-1,2}), \ldots, (u_{1t}, u_{2t}, \ldots, u_{m-1,t}) \tag{9}$$

the points which characterised t edges of the domain $\mathcal{B}_h$, one will determine any point (u) of this domain by the equalities

$$u_i = \sum_{k=1}^{t} \rho_k u_{ik} \quad \text{where} \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m - 1) \tag{10}$$

One will make correspond to the points (9) the points

$$(\xi_r) = (\xi_{1r}, \xi_{2r}, \ldots, \xi_{mr}), \quad (r = 1, 2, \ldots, t) \tag{11}$$

by determining them with the help of equalities (6) and (9).

The points obtained (11) characterise t edges of the domain R belonging to the face P_h. Any point (x) belonging to the face P_h will be determined, on the grounds of (6) and (10), by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{12}$$

Let us notice that all the points (11) verify the equation

$$y_h(x) = 0 \tag{13}$$

and satisfy the conditions

$$y_k(x) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One obtains thus the equalities

$$y_h(\xi_r) \geq 0 \quad (r = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma) \tag{14}$$

The face P_h being in $m - 1$ dimensions, the equalities (14) would define the coefficients of the equation (13) to a close by common factor.

11 Let us suppose that one had determined this way all the faces

$$P_1, P_2, \ldots, P_\sigma \tag{15}$$

in $m - 1$ dimensions of the domain R.

Let us suppose that the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}) \ (k = 1, 2, \ldots, s) \tag{16}$$

characterise the various edges of the domain R belonging to the various faces (15).

By indicating

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \ (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m) \tag{17}$$

one obtains a set of points which all belong to the domain R.

I say that any point (x) belonging to the domain R can be determined with the help of equalities (17).

One can suppose that the point (x) does not belong to any one of the faces (15), since any point belonging to them can be determined with the help of equalities (12).

By supposing that one had the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

let us arbitrarily choose a point (ξ_r) among those of the series (16) and let us admit

$$x'_i = x_i - \rho \xi_{ir} \text{ where } \rho > 0. \ (i = 1, 2, \ldots, m) \tag{18}$$

So long as the parameter ρ is sufficiently small, one will also have

$$y_k(x') > 0. \quad (k = 1, 2, \ldots, \sigma)$$

By making the parameter increase in a continuous manner, one will determine with the help of equalities (18) a point (x') verifying an equation

$$y_h(x') = 0$$

and satisfying the condition

$$y_k(x') \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

The point obtained (x') belongs to the face P_h, therefore one can declare

$$x'_i = \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho'_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (18), it becomes

$$x_i = \rho \xi_{ir} + \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho > 0, \ \rho'_k \geq 0. \ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

It remains to demonstrate that the domain F is in m dimensions.

Let us notice that all the points determined by the equalities (17) with condition

$$\rho_k > 0 \quad (k = 1, 2, \ldots, s)$$

are interior to the domain R.

In effect, all the points (16) verify the inequalities

$$y_h(\xi_k) \geq 0. \ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma) \tag{19}$$

By multiplying these inequalities by ρ_k, let us make the sum of inequalities obtained; one will have, because of (17),

$$y_h(x) = \sum_{k=1}^{s} \rho_k y_h(\xi_k) \geq 0. \quad (h = 1, 2, \ldots, \sigma)$$

By virtue of (19), one will have the inequality

$$y_h(x) > 0, \quad (h = 1, 2, \ldots, s) \tag{20}$$

so long as the numbers $y_h(\xi_1), y_h(\xi_2), \ldots, y_h(\xi_s)$ do not cancel each other out.

One can not suppose that the equalities

$$y_h(\xi_k) = 0 \quad (k = 1, 2, \ldots, s)$$

holds, because otherwise all the equations

$$y_1(x) = 0, y_2(x) = 0, \ldots, y_\sigma(x) = 0$$

would be of proportional coefficients, which is contrary to the hypothesis; therefore one will have the inequalities (20), and it follows that the domain R is of m dimensions.

We have demonstrated that the condition (Ξ) is sufficient for the domain R to be of m dimensions. It is easy to demonstrate that this condition is necessary.

12

We have defined in Number 10 the faces in $m - 1$ dimensions of the domain R. This definition can be generalised.

Definition. One will call face in μ dimensions of the domain R ($\mu = 1, 2, \ldots, m - 1$) a domain $P(\mu)$ formed from points belonging to the domain R and verifying a system of equations

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, \tau) \tag{21}$$

provided that these equations define a domain in μ dimensions composed of points which, all, do not verify any other equation $y_{\tau+1}(x) = 0, \ldots, y_\sigma(x) = 0$.

Let us choose among the points (16) all those which verify the equations (21).

By indicating with

$$\xi_k = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}), \quad (k = 1, 2, \ldots, t)$$

one will declare

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{22}$$

It is easy to demonstrate that any point (x) belonging to the face $P(\mu)$ can be determined with the help of equalities (22).

Corollary. Each face of the domain R is a set of points determined by the equalities (22) provided that any point belonging to it could not be determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

unless all the parameters $\rho_{t+1}, \rho_{t+2}, \ldots, \rho_s$ do not cancel each other.

13

Any point belonging to the domain R either is interior to the domain R or is interior to a face of that domain.

Let us suppose that the point (x) be interior to a face $P(\mu)$ of the domain R which is formed from all the points determined by the equalities (22).

I argue that one can always determine the point (x) by the equalities (22) provided that

$$\rho_k > 0. \quad (k = 1, 2, \ldots, t)$$

To demonstrate this, let us indicate

$$\rho_i = \sum_{k=1}^{t} \xi_{ik}. \quad (i = 1, 2, \ldots, m)$$

The point (α) is interior to the face $P(\mu)$.

By admitting

$$x_i' = x_i - \rho\alpha_i \text{ where; } \rho > 0, \ (i = 1, 2, \ldots, m) \tag{23}$$

one obtains a point (x_i') which will be interior to the face $P(\mu)$ so long as the parameter ρ will be sufficiently small; it follows that

$$x_i' = \sum_{k=1}^{t} \rho_k' \xi_{ik} \quad \text{where} \quad \rho_k' \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (23), one obtains

$$x_i = \sum_{k=1}^{t} (\rho + \rho_k')\xi_{ik}, \quad (i = 1, 2, \ldots, m)$$

and by making

$$\rho + \rho_k' = \rho_k, \quad (k = 1, 2, \ldots, t)$$

one will have

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k > 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

Let us notice that by making $\mu = m$ and $t = s$, one will indicate with the symbol $P(m)$ the domain R; one concludes that any point (x) which is interior to the domain R can be determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k > 0. \quad (k = 1, 2, \dots, s; i = 1, 2, \dots, m)$$

14 *On the correlative domains.*

Definition. Let us suppose that a domain R be determined with the help of inequalities

$$p_{1k}x_1 + p_{2k}x_2 + \dots + p_{mk}x_m \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

One will call correlative to the domain R the domain $\mathcal{R}$ which is formed from all the points (x) determined by the equalities

$$x_i = \sum_{k=1}^{\sigma} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \dots, \sigma; i = 1, 2, \dots, m) \tag{1}$$

I say that the domain $\mathcal{R}$ will be in m dimensions, if the domain R does not possess points verifying the equations

$$p_{1k}x_1 + p_{1k}x_2 + p_{mk}x_m = 0, \quad (k = 1, 2, \dots, \sigma)$$

the point $x_1 = 0, x_2 = 0, \dots, x_m = 0$ being excluded.

In effect, if all the points of the domain $\mathcal{R}$ verified the same equation

$$\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_m x_m = 0,$$

one would have the equalities

$$\xi_1 p_{1k} + \xi_2 p_{2k} + \dots + \xi_m p_{mk} = 0, \quad (k = 1, 2, \dots, \sigma)$$

by virtue of (1), which is contrary to the hypothesis.

Theorem. By supposing that the domain R be formed from all the points (x) determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \dots, s; i = 1, 2, \dots, m) \tag{2}$$

one will define the correlative domain $\mathcal{R}$ with the help of inequalities.

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \dots + \xi_{mk}x_m \geq 0. \quad (k = 1, 2, \dots, s) \tag{3}$$

Let us indicate by $\mathcal{R}'$ thedomain determined with the help of inequalities (3).

On the ground of the supposition made, all the points

$$(\xi_{11}, \xi_{21}, \dots, \xi_{m1}), (\xi_{12}, \xi_{22}, \dots, \xi_{m2}), \dots, (\xi_{1s}, \xi_{2s}, \dots, \xi_{ms})$$

characterise the edges of the domain R, and one will have the inequalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \dots + p_{mh}\xi_{mk} \quad (h = 1, 2, \dots, \sigma; h = 1, 2, \dots, \sigma). \tag{4}$$

We have seen in Number 10 that each face P_h in $m-1$ dimensions of the domain R is characterised by the points

$$(\xi_{11}, \xi_{21}, \dots, \xi_{m1}), (\xi_{12}, \xi_{22}, \dots, \xi_{m2}), \dots, (\xi_{1t}, \xi_{2t}, \dots, \xi_{mt})$$

which verify the equation

$$\overset{x}{y_h} = 0 \tag{5}$$

of the face P_h. One obtains the equalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \dots + p_{mh}\xi_{mk} = 0 \quad (k = 1, 2, \dots, t)$$

which define the coefficients $p_{1h}, p_{2h}, \dots, p_{mh}$ of the equation (5) to an immediate common factor.

One concludes, by virtue of the definition established in Number 9, that the point $(p_{1h}, p_{2h}, \dots, p_{mh})$ characterises an edge of the domain $\mathcal{R}'$.

By attributing with the indice h the values $1, 2, \dots, \sigma$, one obtains a series

$$(p_{11}, p_{21}, \dots, p_{m1}), (p_{12}, p_{22}, \dots, p_{m2}), \dots, (p_{1\sigma}, p_{2\sigma}, \dots, p_{m\sigma})$$

of points which characterise different edges of the domain $\mathcal{R}'$.

I argue that the domain $\mathcal{R}'$ does not possess other edges. To demonstrate this, let us suppose that a point $p_1, p_2, \dots, p_m$ characterises an edge of the domain $\mathcal{R}'$.

One will have the equalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \dots + p_m\xi_{mh} = 0, \quad (h = 1, 2, \dots, t) \tag{6}$$

which define the coefficients $p_1, p_2, \dots, p_m$ to a nearby common factor, and one will have the inequalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \dots + p_m\xi_{mh} \geq 0. \quad (k = 1, 2, \dots, s) \tag{7}$$

Let (x) be any point of the domain R. One will determine the point (x) with the help of equalities (2). By multiplying the inequalities (7) with ρ_k and by making the sum of inequalities obtained, one will have, because of (2),

$$p_1x_1 + p_2x_2 + \dots + p_mx_m \geq 0.$$

One concludes that the inequalities

$$-p_1x_1 - p_2x_2 - \ldots - p_mx_m \geq 0 \text{ and } p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0,$$

define a domain which is not in m dimensions.

By virtue of the fundamental theorem of Number 10, one will determine in this case positive values or zeros of parameters $\rho, \rho_1, \ldots, \rho_\sigma$ which reduce the equation

$$-\rho(p_1x_1 + p_2x_2 + \ldots + p_mx_m) + \sum_{k=1}^{\sigma} \rho_k(p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m) = 0$$

into an identity.

It follows that

$$p_i = \sum_{k=1}^{\sigma} \frac{\rho_k}{\rho} p_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0. \quad (k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, m)$$

By substituting (6), one will have

$$\sum_{k=1}^{\sigma} \frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk}) = 0. \quad (h = 1, 2, \ldots, t)$$

By virtue of (4), one finds

$$\frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk}) = 0. \ (h = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma)$$

Let us suppose that $\frac{\rho_k}{\rho} > 0$, then

$$\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk} = 0, \quad (h = 1, 2, \ldots, t)$$

therefore the coefficients $p_1, p_2, \ldots, p_m$, by virtue of (6), are proportional to the coefficients $p_{1k}, p_{2k}, \ldots, p_{mk}$; it follows that the points $(p_{1k}, p_{2k}, \ldots, p_{mk})$ and $(p_1, p_2, \ldots, p_m)$ characterise the same edge of the domain $\mathcal{R}'$.

By virtue of the fundamental theorem in Number 10, all the points of the domain $\mathcal{R}'$ will be determined by the equality (1), this results in that the domains $\mathcal{R}$ and $\mathcal{R}'$ coincide.

Corollary. Let us suppose that a face $P(\mu)$ in μ dimensions of the domain R be determined by the equations

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m = 0, \quad (k = 1, 2, \ldots, \tau)$$

and that any point (x) belonging to the face $P(\mu)$ be determined by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

The correlative domain $\mathcal{R}$ will possess a corresponding face $\mathcal{B}(m - \mu)$ in $m - \mu$ dimensions determined by the equations

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m = 0 \quad (k = 1, 2, \ldots, t)$$

and any point (x) belonging to the face $\mathcal{B}(m - \mu)$ will be determined by the equalities

$$x_i = \sum_{k=1}^{\tau} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, \tau; i = 1, 2, \ldots, m)$$

Definition of domains of quadratic forms corresponding to the various perfect forms

15 Let us consider any one perfect quadratic form φ.

Let us suppose that all the representations of the minimum of the form φ make up the series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \tag{1}$$

By indicating

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n, \quad (k = 1, 2, \ldots, s) \tag{2}$$

one corresponds to the series (1) a series of linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s.$$

Let us consider a domain R of quadratic forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{s} \rho_k \lambda_k^2$$

with condition that

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, s)$$

One will say that the domain R correspond to the perfect form φ.

Let us notice that the domain R is in $\frac{n(n+1)}{2}$ dimensions.

By supposing the contrary let us suppose that all the quadratic forms belonging to the domain R verifies a linear equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0.$$

On the ground of the established definition, one will have the equalities

$$\Psi(\lambda_k^2) = 0 \quad (k = 1, 2, \ldots, s)$$

or, that which comes to the same thing, because of (2),

$$p_{ij} l_{ik} l_{jk} = 0 \quad (k = 1, 2, \ldots, s)$$

which is impossible, the form φ being perfect.

On the ground of what has been said in Number 9–14, the domain R possesses s edges characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_s^2. \tag{3}$$

Let us suppose that one had determined all the faces

$$P_1, P_2, \ldots, P_\sigma$$

in $\frac{n(n+1)}{2} - 1$ dimensions of the domain R.

Each face P_k can be determined by two methods:

1. All the quadratic forms belonging to the face P_k verify an equation

$$\Psi_k(f) = \sum p_{ij}^{(k)} a_{ij} = 0$$

which can be determined in such a way that the inequality

$$\Psi_k(f) > 0$$

held so long as the form f belonging to the domain R is exterior to the face P_k.

2. By choosing among the quadratic forms (3) these

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2$$

which verify the equation (4), one will determine all the quadratic forms belonging to the face P_k by the equalities

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2,$$

where

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, t)$$

By virtue of the theorem of Number 14, the domain R can be considered as a set of points verifying the inequalities

$$\Psi_k(f) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

On the extreme quadratic forms

16 Let us indicate by $M(a_{ij})$ the minimum and by $D(a_{ij})$ the determinant of a positive quadratic form $\sum a_{ij} x_i x_j$. The positive quadratic form $\frac{1}{\sqrt[n]{D(a_{ij})}} \sum a_{ij} x_i x_j$ will be of determinant 1 and will possess the minimum

$$\frac{M(\alpha_{ij})}{\sqrt[n]{D(a_{ij})}} = \mathcal{M}(a_{ij}).$$

Let us examine the various value of the function $\mathcal{M}(a_{ij})$ which is well determined in the set (f) of all the positive quadratic forms in n variables.

Definition. One will call extreme ‡ *a positive quadratic form* $\sum a_{ij} x_i x_j$ *which enjoys the property that the corresponding value of the function* $\mathcal{M}(a_{ij})$ *is minimum.*

Let us notice that the function $\mathcal{M}(a_{ij})$ does not change its value when one replaces quadratic form $\sum a_{ij} x_i x_j$ by a form of proportional coefficients.

By attributing to the coefficients of the extreme form $\sum a_{ij} x_i x_j$ variations

$$\epsilon_{ij} = \epsilon_{ji} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

satisfying the condition

$$|\epsilon_{ij}| < \epsilon, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{1}$$

ϵ being an arbitrary positive parameter, let us examine the corresponding value of the function $\mathcal{M}(a_{ij})$.

On the ground of the definition established, one can determine the parameter ϵ such that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) < \mathcal{M}(a_{ij}) \tag{2}$$

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‡ See the Mémoire of Mr.'s Korkine and Zolotareff, Sur les formes quadratiques [On the quadratic forms], Mathematische Annalen V. VI, p. 368

held with condition (1) and so long as the coefficients ϵ_{ij} are not proportional to the coefficients

$$a_{ij}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

17 *Theorem. For a quadratic form* $\sum a_{ij}x_ix_j$ *to be extreme, it is necessary and sufficient that it be perfect and that its adjointed form* $\sum \frac{\partial D(a_{ij})}{\partial a_{ij}} x_ix_j$ *be interior to the domain corresponding to the form* $\sum a_{ij}x_ix_j$.

Let us indicate by

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}) \tag{3}$$

the various representations of the minimum $M(a_{ij})$ of the form $\sum a_{ij}x_ix_j$.

Let us consider a quadratic form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$, the parameter ρ being arbitrary. One can determine an interval

$$-\delta < \rho < \delta \text{ where } 0 < \delta < 1 \tag{4}$$

such that all the representations of the minimum of the form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ are found among the systems (3) so long as the variations ϵ_{ij} satisfy the condition (1).

By indicating with

$$M' = \sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk} \text{ and } M = \sum a_{ij}l_{ik}l_{jk} \tag{5}$$

the minima of forms $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ and $\sum a_{ij})x_ix_j$ and with D' and D their determinants, one will have

$$\mathcal{M}(a_{ij} + \rho\epsilon_{ij}) = \frac{\sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}}, \quad \mathcal{M}(a_{ij}) = \frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}.$$

By virtue of (2), one obtains the inequality

$$\frac{\sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}} < \frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}$$

or, that which comes to the same thing,

$$\rho\sum \epsilon_{ij}l_{ik}l_{jk} < M\left(\sqrt[n]{\frac{D'}{D}} - 1\right). \tag{6}$$

This declared, let us suppose that the form $\sum a_{ij}x_ix_j$ be not perfect.

One will determine in this case the variations ϵ_{ij} such that the equalities

$$\sum \epsilon_{ij}l_{ik}l_{jk} = 0,$$

held. By virtue of (6), one will obtain the inequality

$$D' > D.$$

By developping the determinant D' into a series, one will have the inequality

$$\rho\sum \epsilon_{ij}\frac{\partial D}{\partial a_{ij}} + \frac{\rho^2}{2}\sum \epsilon_{ij}\epsilon_{kh}\frac{\partial^2 D}{\partial a_{ij}\partial a_{kh}} + \ldots > 0. \tag{7}$$

The parameter ρ being arbitrary satisfying the condition (4), it is necessary tht

$$\sum \epsilon_{ij}\frac{\partial D}{\partial a_{ij}} = 0.$$

Mr.'s Korkine and Zolotareff have demonstrated ‡ that in this case one will always have the inequality

$$\sum \epsilon_{ij}\epsilon_{kh}\frac{\partial^2 D}{\partial a_{ij}\partial a_{kh}} < 0,$$

therefore the inequality (7) is impossible.

18 We have demonstrated that the form $\varphi = \sum a_{ij}x_ix_j$ has to be perfect.

Let us suppose that the domain R corresponding to the perfect form φ be determined by σ inequalities

$$\Psi_r(f) = \sum p_{ij}^{(r)}a_{ij} \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

On the grounds of these inequalities, one will have

$$\Psi_r(\lambda_k^2) = \sum p_{ij}^{(r)}l_{ik}l_{jk} \geq 0. \quad (k = 1, 2, \ldots, s; r = 1, 2, \ldots, \sigma) \tag{8}$$

Let us declare

$$\epsilon_{ij} = tp_{ij}^{(r)} \text{ where } t > 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

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‡ Mathematische Annalen, V. XI, p. 250

By virtue of (6), one will have

$$\rho t \sum p_{ij}^{(r)} l_{ik} l_{jk} < M \left(\sqrt[n]{\frac{D'}{D}} - 1 \right). \tag{9}$$

Let us attribute to the parameter ρ a positive value satisfying the condition (4), by virtue of (8) and (9) there will arrive

$$D' > D.$$

By developping the determinant D' into a series, one obtains the inequality

$$\rho t \sum p_{ij}^{(r)} \frac{\partial D}{\partial a_{ij}} + \frac{(\rho t)^2}{2} \sum p_{ij}^{(r)} p_{ih}^{(r)} \frac{\partial^2 D}{\partial a_{ij} \partial a_{kh}} + \ldots > 0.$$

The positive parameter ρ being as small as one wish, it follows that

$$\sum p_{ij}^{(r)} \frac{\partial D}{\partial a_{ij}} > 0. \quad (r = 1, 2, \ldots, \sigma)$$

It is thus demonstrated that the form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjointed to the form φ, is interior to the domain R.
I argue that in this case the perfect form φ will be extreme.
By supposing the contrary, let us suppose that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) \geq \mathcal{M}(a_{ij}) \tag{10}$$

be verified by any one system of variations ϵ_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ satisfying the condition (1) however small the parameter ϵ may be.
By virtue of (10), one obtains

$$\sum \epsilon_{ij} l_{ik} l_{jk} \geq M \left(\sqrt[n]{\frac{D'}{D}} - 1 \right); \quad (k = 1, 2, \ldots, s) \tag{11}$$

the inequality obtained has to hold whatever may te the value of the index $k = 1, 2, \ldots, s$.
By indicating

$$\eta_{ij} = a_{ij} \left(\sqrt[n]{\frac{D}{D'}} - 1 \right) + \epsilon_{ij} \sqrt[n]{\frac{D}{D'}}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{12}$$

let us examine the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \sum (a_{ij} + \eta_{ij}) x_i x_j. \tag{13}$$

By virtue of (12) the form φ_0 is of determinant D.
By choosing the parameter ϵ sufficiently small, one can suppose that

$$|\eta_{ij}| < \eta, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{14}$$

η being a positive parameter as small as one would like.
By virtue of (5), (11) and (12), one obtains

$$\sum \eta_{ij} l_{ik} l_{jk} \geq 0. \quad (k = 1, 2, \ldots, s) \tag{15}$$

By developping the determinant D of the form (13) into series, one will find

$$D + \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} + R_2 = D. \tag{16}$$

In this equality the remainder R_2 verifies an inequality

$$|R_2| < \eta^2 P,$$

P being a positive number not depending on the parameter η so long as $\eta < 1$.
By virtue of (16), one obtains

$$\left| \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} \right| < \eta^2 P. \tag{17}$$

We have suppose that the quadratic form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjointed to the form φ, be interior to the domain R. On the ground of that which has been said in Number 13, one will determine the form $\sum \frac{\partial D}{\partial a_{ij}} \cdot x_i x_j$ with the help of the equality

$$\sum \frac{\partial D}{\partial a_{ij}} x_i x_j = \sum_{k=1}^{s} \rho_k \lambda_k^2 \tag{18}$$

where

$$\rho_k > 0. \quad (k = 1, 2, \ldots, s) \tag{19}$$

The equality (18) can be replaced by the following ones:

$$\frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k l_{ik} l_{jk}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By multiplying these equations by η_{ij} and by adding up the equalities obtained, one will have

$$\sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k \sum \eta_{ij} l_{ik} l_{jk}. \tag{20}$$

By virtue of (15), (17) and (19), one obtains the inequalities

$$0 \leq \sum \eta_{ij} l_{ik} l_{jk} < \eta^2 \frac{P}{\rho_k}; \quad (k = 1, 2, \ldots, s)$$

therefore one can admit

$$\sum \eta_{ij} l_{ik} l_{jk} = \tau_k \eta^2, \quad (k = 1, 2, \ldots, s) \tag{21}$$

and the positive numbers or zeros τ_k $(k = 1, 2, \ldots, s)$ will not exceed fixed limits which do not depend on the parameter η.

After the definition of perfect forms, the equations (21) admit only a single system of solutions. By effecting this solution of equations (21), one obtains

$$\eta_{ij} = \tau_{ij} \eta^2 \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where

$$|\tau_{ij}| < T, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

T being a positive number which does not depend on the parameter η; therefore one will have the inequalities

$$|\eta_{ij}| < \eta^2 T. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{22}$$

This stated, let us take any one positive fraction ϑ and declare

$$\eta = \frac{\vartheta}{T}.$$

By virtue of (14), one will have

$$|\eta_{ij}| < \frac{\vartheta}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and because of (22), it will become

$$|\eta_{ij}| < \frac{\vartheta^2}{T}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By admitting

$$\eta_{ij} < \frac{\vartheta^2}{T},$$

one will have, because of (22),

$$|\eta_{ij}| < \frac{\vartheta^4}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and so on.

One will obtain in this manner the inequalities

$$|\eta_{ij}| < \frac{\vartheta^{2^k}}{T} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; k = 0, 1, 2, \ldots)$$

it follows that

$$\eta_{ij} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By virtue of (12), one obtains

$$\epsilon_{ij} = a_{ij} \left(\sqrt[n]{\frac{D'}{D}} - 1 \right); \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

therefore the coefficients ϵ_{ij} are proportional to the coefficients a_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$, which is contrary to the hypothesis.

Properties of the set of domains corresponding to the various perfect forms in n variables.

19 Any perfect form φ will be transformed by an equivalent perfect form φ' with the help of any substitution S of integer coefficients and of determinant ±1.

Let us indicate by R and R' the domains corresponding to the perfect forms φ and φ' and by T the substitution adjointed to the substitution S.

One will easily demonstrate that the domain R will be transformed into an equivalent domain R' with the help of the substitution T.

One concludes that there exists a finite number of domains equivalent to the domain R.

Let us indicate by (R) the set of all the domains corresponding to the various perfect forms in n variables.

The set (R) can be divided into classes of equivalent domains.

On the ground of that which has been previously said, the number of different classes of the set (R) is equal to the number of classes of perfect forms in n variables.

20 *Theorem. Let us suppose that a quadratic form f be interior to a face $P(\mu)$ in μ dimensions of the domain R $(\mu = 1, 2, \dots, \frac{n(n+1)}{2})$.*

The form f will belong only to the domains of the set (R) which are contiguous through the face $P(\mu)$.

Let us suppose that the domain R be characterised by the quadratic form

$$\lambda_1^2, \lambda_2^2, \dots, \lambda_s^2 \tag{1}$$

and that the face $P(\mu)$ in μ dimensions of the domain R be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \dots, \lambda_t^2. \tag{2}$$

In the case $\mu = \frac{n(n+1)}{2}$, one will admit $t = s$, and the symbol $P\left(\frac{n(n+1)}{2}\right)$ will indicate the domain R.

The quadratic form f being interior to the face $P(\mu)$, one will have the equality

$$f(x_1, x_2, \dots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2 \text{ where } \rho_k > 0. \ (k = 1, 2, \dots, t) \tag{3}$$

Let us suppose that the same form f belonged to another domain R' of the set (R).

Let us suppose that the domain R' be characterised by the quadratic forms

$${\lambda'_1}^2, {\lambda'_2}^2, \dots, {\lambda'_\sigma}^2 \tag{4}$$

and that the form f be interior to the face $P'(\nu)$ of the domain R' characterised by the quadratic forms

$${\lambda'_1}^2, {\lambda'_2}^2, \dots, {\lambda'_\tau}^2. \tag{5}$$

One will have, on the ground of the supposition made,

$$f(x_1, x_2, \dots, x_n) = \sum_{h=1}^{\tau} \rho'_h {\lambda'_h}^2 \text{ where } \rho'_h > 0. \ (h = 1, 2, \dots, \tau) \tag{6}$$

This declared, let us indicate by φ and φ' the perfect forms corresponding to the domains R and R' and suppose, for more simplicity, that the minimum of forms φ and φ' be M.

By indicating with the symbol (f, f') the result

$$(f, f') = \sum a_{ij} a'_{ij},$$

from two quadratic forms

$$f(x_1, x_2, \dots, x_n) = \sum a_{ij} x_i x_j \text{ and } f'(x_1, x_2, \dots, x_n) = \sum a'_{ij} x_i x_j,$$

let us examine two results (f, φ) and (f, φ').

By virtue of (13), one obtains

$$(f, \varphi) = \sum_{k=1}^{t} \rho_k (\varphi, \lambda_k^2) \text{ and } (f, \varphi') = \sum_{k=1}^{t} \rho_k (\varphi', \lambda_k^2). \tag{7}$$

By virtue of (6), one obtains

$$(f, \varphi) = \sum_{h=1}^{\tau} \rho'_h (\varphi, {\lambda'_h}^2) \text{ and } (f, \varphi') = \sum_{h=1}^{\tau} \rho'_h (\varphi', {\lambda'_h}^2). \tag{8}$$

Let us notice that

$$(\varphi, \lambda_k^2) = M \text{ and } (\varphi', \lambda_k^2) \geq M; \ (k = 1, 2, \dots, s) \tag{9}$$

$$(\varphi, {\lambda'_h}^2) \geq M \text{ and } (\varphi', {\lambda'_h}^2) = M. \ (h = 1, 2, \dots, \sigma) \tag{10}$$

From equalities (7), one derives

$$(f, \varphi') - (f, \varphi) = \sum_{k=1}^{t} \rho_k \left[(\varphi', \lambda_k^2) - (\varphi, \lambda_k^2) \right], \tag{11}$$

and by virtue of (3) and (9) there comes

$$(f, \varphi') - (f, \varphi) \geq 0.$$

From equalities (8), one derives

$$(f,\varphi') - (f,\varphi) = \sum_{h=1}^{\tau} \rho'_h \left[(\varphi', {\lambda'_h}^2) - (\varphi, {\lambda'_h}^2)\right], \tag{12}$$

and by virtue of (6) and (10), one will have

$$(f,\varphi') - (f,\varphi) \leq 0.$$

It follows that

$$(f,\varphi') = (f,\varphi),$$

and the equalities (11) and (12) give

$$(\varphi', \lambda_k^2) = (\varphi, \lambda_k^2), \quad (k = 1, 2, \ldots, t)$$

$$(\varphi', {\lambda'_h}^2) = (\varphi, {\lambda'_h}^2). \quad (h = 1, 2, \ldots, \tau)$$

By virtue of (9) and (10), there arrive

$$(\varphi, {\lambda'_h}^2) = M, \quad (h = 1, 2, \ldots, \tau) \tag{13}$$

$$(\varphi', \lambda_k^2) = M. \quad (k = 1, 2, \ldots, t) \tag{14}$$

By virtue of equalities (13), the quadratic forms (5) are found among those of the series (1). By virtue of (14), the quadratic forms (2) are found among those of the series (4).

I argue that in this case the series (2) and (5) contain the same forms.

To demonstrate this, let us suppose that all the forms belonging to the face $P(\mu)$ verify the equations

$$\Psi_1(f) = 0, \ \Psi_2(f) = 0, \ldots, \ \Psi_r(f) = 0$$

and that any form belonging to the domain R verifies the inequalities

$$\Psi_1(f) \geq 0, \ \Psi_2(f) \geq 0, \ldots, \ \Psi_r(f) \geq 0. \tag{15}$$

By virtue of (6), one will have

$$\rho'_1 \Psi_i({\lambda'_1}^2) + \rho'_2 \Psi_i({\lambda'_2}^2) + \ldots + \rho'_\tau({\lambda'_\tau}^2) = 0, \quad (i = 1, 2, \ldots, r)$$

and because of (15), one finds

$$\Psi_i({\lambda'_h}^2) = 0; \quad (i == 1, 2, \ldots, r; h = 1, 2, \ldots, \tau)$$

therefore all the forms of the series (5) belong to the series (2).

In the same way, one will demonstrate that all the forms of the series (2) belong to the series (5).

One concludes that the faces $P(\mu)$ and $P'(\nu)$ coincide, therefore the domains R and R' are contiguous through the face $P(\mu)$.

Corollary. A quadratic form which is interior to a domain of the set (R) *can not belong to any other domain of that set.*

21 *Theorem. Let us suppose that to a face* $P(\mu)$ *of the domain* R *belong positive quadratic forms. In this case, the number of domains of the set* (R) *contiguous through the face* $P(\mu)$ *is finite.*

Let us indicate by

$$R, R_1, R_2, \ldots$$

the domains of the set (R) contiguous through the face $P(\mu)$. Let

$$\varphi, \varphi_1, \varphi_2, \ldots$$

be the corresponding perfect forms having the minimum M.

On the ground of the supposition made, one positive quadratic form f will be interior to the face $P(\mu)$.

We have demonstrated in the previous number that

$$(f,\varphi) = (f,\varphi_1) = (f,\varphi_2) = \ldots \tag{16}$$

It is easy to demonstrate that the number of perfect forms having the minimum M and verifying the equalities (16) is finite.

Algorithm for the search for the domain of the set (R) *contiguous to another domain by a face in* $\frac{n(n+1)}{2} - 1$ *dimensions*

22 Let

$$\varphi(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$$

be a perfect form having the minimum M the various representation of which make up a series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \tag{1}$$

Let us suppose that a face P of the domain R corresponding to the perfect form φ be determined by the equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0$$

and by the condition

$$\Psi(f) \geq 0$$

which is verified by any quadratic form belonging to the domain R.

Let us suppose that the face P be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2 \tag{2}$$

where

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n. \quad (k = 1, 2, \ldots, s)$$

On the ground of the supposition made, one will have the equalities

$$\sum p_{ij} l_{ik} l_{jk} = 0 \ \ (k = 1, 2, \ldots, t) \tag{3}$$

which define the coefficients P_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ to an immediate common factor.

Let us suppose that the face P could belong to the other domains of the set (R). Let us indicate by R' a similar domain. Let φ' be the perfect form corresponding to the domain R'.

By virtue of the supposition made, the quadratic form (2) belong to the domains R and R'. It results in that the systems

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, [\ldots,]l_{n2}), \ldots, (l_{1t}, l_{2t}, \ldots, l_{nt}) \tag{4}$$

corresponding to the forms (2) represent the minimum of forms φ and φ'.

Let us suppose, for more simplicity, that the forms φ and φ' had the minimum M. One will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M \ \text{ and } \ \sum a'_{ij} l_{ik} l_{jk} = M, \ \ (k = 1, 2, \ldots, t) \tag{5}$$

by putting

$$\varphi'(x_1, x_2, \ldots, x_n) = \sum a'_{ij} x_i x_j.$$

From equation (5), one gets

$$\sum (a'_{ij} - a_{ij}) l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, t)$$

and by virtue of (3), it becomes

$$a'_{ij} = a_{ij} + \rho p_{ij}. \ \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{6}$$

Let us indicate

$$\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j.$$

By virtue of (16), one obtains

$$\varphi'(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n). \tag{7}$$

This stated, let us choose in the series (1) a system $(l_{1h}, l_{2h}, \ldots, l_{nh})$ which does not belong to the series (4). As

$$\varphi(l_{1h}, l_{2h}, \ldots, l_{nh}) = M \quad \varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) \geq M$$

and

$$\Psi(l_{1h}, l_{2h}, \ldots, l_{nh}) > 0,$$

one deduces from the equality

$$\varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) = \varphi(l_{1h}, l_{2h}, \ldots, l_{nh}) + \rho\Psi(l_{1h}, l_{2h}, \ldots, l_{nh})$$

the inequality

$$\rho \geq 0.$$

The supposition $\rho = 0$ being obviously impossible, one obtains

$$\rho > 0,$$

and it follows that

$$\varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) > M.$$

Let us indicate by

$$(l'_1, l'_2, \ldots, l'_n), (l''_1, l''_2, \ldots, l''_n), \ldots, (l_1^{(r)}, l_2^{(r)}, \ldots, l_n^{(r)}) \tag{8}$$

all the representations of the minimum of the perfect form φ' which are not found in the series (4). By virtue of (7), one will have

$$\varphi'(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) = \varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) + \rho\Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) \qquad (\text{k} = 1, 2, \ldots, \text{r})$$

which results in

$$\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) > M \ \text{ and } \ \Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) < 0. \qquad (k = 1, 2, \ldots, r) \tag{9}$$

The value of the parameter ρ will have for expression

$$\rho = \frac{\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) - 1}{-\Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)})}. \ \ (k = 1, 2, \ldots, r)$$